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Working with mathematically gifted students

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Abstract

Working with gifted students is of special social interest. However, in the existing secondary education curricula, it is only declaratively mentioned and this important segment of the education system is completely neglected. That is why in this paper an attempt was made to develop a program for work with mathematically gifted students aged 17–18, that is, for the third year of secondary education, using the concentric circle method. This study uses a descriptive method and gives examples of contents for the curriculum for working with gifted students. Curriculum materials were prepared using the method of concentric circles, which means that some contents adopted in previous years at a certain level are extended and expanded. The study draws from resources from previous researchers and existing literature.

Keywords: Concentric circle; curricula; gifted students; mathematics.

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1. Introduction

Gogovska et al. (2021) and Anevska et al. (2022) provide the curricula for working with mathematically gifted students from the first and second years of secondary education. Having in mind the social significance of working with mathematically gifted students and the fact that as a rule it is realised turbulently, in this paper we will provide an integral curriculum for working with mathematically gifted students aged 17–18 years. We believe that the preparation of such a curriculum, as well as the preparation of appropriate books and collections of problems that are complementary to the curriculum, will greatly fill the gap caused by society's neglect of the development of these children.

In almost all countries, without exception, there is formal support for the progress of these children, most often expressed through the organisation of competitions and awarding of scholarships, and practically doing nothing further for the development of these children (Attar, 2019; Cetinkayaa & Cetinkaya, 2020; Katanani & Sakarneh, 2021).

1.1. Purpose of study

This paper is in a way a continuation of the above-mentioned papers. In addition, based on the experience of the authors, but also the experience of the countries in the immediate and wider surrounding, an attempt was made for part of the topic geometry of a complex number to give an example of a system of problems that would determine the level that students should reach this age.

2. Materials and methods

This study uses a descriptive method to analyse the best curriculum for working with gifted students. Curriculum materials were prepared using the method of concentric circles, which means that some contents adopted in previous years at a certain level are extended and expanded. The study draws from resources from previous researchers and existing literature.

3. Results

3.1. Curriculum for working with mathematically gifted students aged 17–18

In this section, we will present a curriculum for working with mathematically gifted students aged 17–18, that is, for students in the third year of secondary education. The offered curriculum builds on the respective teaching curriculum that was previously prepared for students in secondary education and is presented in Gogovska et al. (2021) and Anevska et al. (2022). During the preparation of the curriculum, the method of concentric circles was used, which means that part of the contents that were adopted in the previous years at a certain level is expanded and extended. This curriculum should be implemented continuously, and not only in periods when students are preparing for certain math competitions. The goals of the curriculum for students aged 17–18 are:

- To develop students' qualities of thinking such as flexibility, stereotyping, width, rationality, depth and criticality (Celik & Yavuz, 2018),
- The student applies the scientific methods: observation, comparison, experiment, analysis, synthesis, classification, systematisation and axiomatic method,
- The student to apply the types of conclusions: induction, deduction and analogy, whereby it is of particular importance to present suitable examples from which the student will realise that the analogy conclusion is not always correct (Keser & Erdem, 2019),
- The student to adopt the prescribed contents in the field of groups, rings, integral domains and fields and to enable them to apply the same when solving appropriate problems,
- The student adopts the prescribed contents in the field of matrices and determinants and enables them to apply the same when solving problems,
- The student is to adopt the prescribed contents in the field of Euclidean vector spaces, scalar, vector and mixed product, as well as the equation of a straight line in a plane and

space, the equation of plane and their relative positions, and to be able to apply the acquired knowledge in problem-solving,

- The student to adopt the prescribed contents in the field of second-order curves and to be able to apply the acquired knowledge in problem-solving,
- The student to adopt the prescribed contents in the field of a trigonometric and exponential form of a complex number and to be able to apply the acquired knowledge in problem-solving,
- The student to adopt the prescribed contents in the field of the geometry of a complex number and to be able to apply the acquired knowledge in problem-solving,
- The student to adopt the prescribed contents in the field of theory of polynomials and to be able to apply the acquired knowledge in problem-solving,
- The student to adopt the prescribed contents in the field of axiomatic construction of the set of real numbers and fields and to be able to apply the acquired knowledge in problem-solving,
- The student to adopt the prescribed contents in the field of the sequences of real numbers and to be able to apply the acquired knowledge in problem-solving, and
- The student is to adopt the prescribed contents in the field of Number theory and to be able to apply the acquired knowledge in problem-solving.

To achieve the aforementioned goals, it is necessary to adopt the following contents:

3.1.1. Linear algebra with analytic geometry (4 classes per week – 144 classes per year)

Algebraic structures: groupoid, semigroup, group: definition and properties, subgroup, Lagrange's theorem, ring, integral domain and field, the field *Z*.

Matrices, determinants, system of linear equations: the equation $ax = b$ in a field, system of field equations, equivalent system of equations, solving a system of equations, Gaussian method, notion of matrix, addition of matrices, multiplying a matrix by a scalar and multiplication of matrices, singular matrix, the notion of inverse matrix, the matrix A^n, examples, notion of determinant, properties of determinants, Sarrus rule, calculation of the determinant, Laplace theorem, inverse matrix, condition of regularity, calculation of inverse matrix, elementary transformations of matrices and their application, matrix rank, basic minor theorem, representing a linear system in matrix form, Kronecker-Capelli theorem, Cramer's rule, other methods for solving a system of linear equations.

Plane analytic geometry: Cartesian rectangular coordinate system in a plane, coordinates of points and vectors, distance between two points, partitioning a line-segment in a given ratio, equation of a line (different types), angle between two lines (condition of parallelism and perpendicularity), relative positions between two lines, distance from a point to a line, circle: notion and properties, relative position of circle and a straight line, equation of the tangent to a circle, ellipse: notion and properties, relative position of ellipse and straight line, equation of the tangent to an ellipse, hyperbola: notion and properties, asymptotes of a hyperbola, relative position of hyperbola and straight line, equation of the tangent to a hyperbola, parabola: notion and properties, relative position of parabola and straight line, equation of the tangent to a parabola, second-order curves, second order curves as conic sections, reducing a second-order curve to canonical form, coordinate transformations.

Analytic geometry of space: Cartesian rectangular coordinate system in space, scalar and vector quantities, projection of vector into an axis, projection of vector on coordinate axis, cosine direction, addition and subtraction of vectors, multiplication of a vector by a scalar, linear dependence and linear independence of vectors, projection theorem, vector decomposition into components, scalar product of vectors, coordinate form of a scalar product, vector product of vectors, coordinate form of vector product, mixed product of vectors, coordinate form of a mixed product, transformations of Cartesian rectangular coordinates, equation of a straight line in a plane, types of equation of straight line in a plane, pencil of lines, perpendicular lines, condition for perpendicularity of two lines, distance from a point to a line in a plane, bisector of an angle between two lines, equation of a plane in space, incomplete equations of a plane, segment form of plane equation, normal form of equation of a plane, distance from a point to a plane, angle between two planes, equation of a straight line in space, relative position of two straight lines, relative position of a line and plane.

3.1.2. Geometry of a complex number (2 classes per week – 72 classes per year)

Trigonometric form of a complex number: trigonometric notation of a complex number, operations and De Moivre's formula, formulas for $\sin^n x$ and $\cos^n x$, *n* is a natural number, roots of complex numbers, solving binomial equations, exponential form of a complex number, Euler's formulas, extended complex plane, Riemann's interpretation of a complex number.

Transformations of the Euclidean plane: equation of a line, perpendicular and parallel lines, distance from a point to a line, equation of a circle, direct similarities, movements, homothety, indirect similarities, inversion, Mobius transformation, and its geometric properties.

Geometry of a circle and a triangle: central and inscribed angle of a circle, Thales' theorem, tangent-chord theorem, power of a point to a circle, radical axis and radical centre, a pencil of circles, orthocentre and centroid of a triangle, Leibniz theorem, midsegment of a triangle (properties), right triangle, Pythagorean theorem, Euler's line, and Euler's circle, Menelaus' theorem, Desargues' theorem, and Pascal's theorem, triangular coordinates, Ceva's theorem, and Van Aubel's theorem, area of a triangle, area of a convex polygon, incircle, and excircles of a triangle, Euler's theorem, Stewart's theorem, length of a median, lengths of the bisectors of interior angles of a triangle, symmedian (properties), Simson line, Ptolemy's theorem, and Ptolemy's inequality, scalar product, Apollonius' theorem, Lagrange's theorem, and Leibniz theorem.

3.1.3. Algebra and analysis (2 classes per week – 76 classes per year)

Polynomials: the notion of polynomial, GCD of polynomials, Euclidean algorithm, zeros of a polynomial, fundamental theorem of algebra (without proof), Vieta's formulas, cubic equation, Cardano's formula, systems of nonlinear equations.

Axiomatic basis of the set of real numbers: real numbers, field axioms, some consequences of the introduced axioms, property of continuity of the set of real numbers, the decimal representation of the real numbers, the density of the sets of rational and irrational numerical numbers, countable sets, uncountability of the set of real numbers.

Sequences: arithmetic sequence, notion, term, and the sum of terms, geometric sequence, notion, term, and the sum of terms, identities, and sums, sequence, definition, and properties, the limit value of a sequence, convergent sequences, properties of convergent sequences, monotone and finite sequences, the convergence of a monotone sequence, the number e, three-series theorem, Cantor and Bolzano-Weierstrass theorems, Cauchy sequence, Stolz theorem and application, infinite order, geometric order.

Number theory: Euler theorem, Fermat's theorem, Carmichael's theorem (revision), order of integers by modulo, primitive roots, Mobius function, cyclotomic polynomials, quadratic residues, Pell's equation, Chebyshev's theorem on the distribution of prime numbers (without proof), Bertrand's postulate, Dirichlet theorem, continued fractions and Farey sequence, Diophantine approximations.

3.2. Example of a system of problems from the section on geometry of a complex number

To realise the suggested curriculum for working with gifted 17–18 years old students, it is necessary to make appropriate teaching aids, that is to say, textbooks that must be accompanied by appropriate books with collections of problems. Hereinafter, we will present a system of problems that we deem suitable for studying the second and third parts of the content of the topic geometry of a complex number.

1. Given two lines (p) and (q) and a point A. Through A draw a line (a), so that A is the midpoint of the line segment MN, where $M=(a)\cap(p)$ and $N=(a)\cap(q)$.

2. Given lines (p) and (q) and a point A. Construct an equilateral triangle ABC, such that $B\in(p)$ and $C\in(q)$.

3. Given lines (p) and (q) and a point O. Construct an equilateral triangle ABC centred at the point O, so that its two vertices are placed on the lines (p) and (q), respectively.

4. Given a circle $K(O,r)$ and a point A on the circle. Determine the locus of midpoints of the chords of the circle (K) at the point A.

5. Let $ABCD$ be trapezoid with bases AB and CD and let M be the midpoint of AB, N be the midpoint of CD, P be the intersection of diagonals and Q the intersection of extensions of the legs. Prove that the points M,N,P and Q are collinear.

6. Prove that if two circles touch each other, their centres and the point of touch are collinear.

7. Given points A and B. Let A' be a point on the line OB, B' be a point on the line OA and Z be a point on AB. Construct a point Z' which divides the line segment $A'B'$ in the same ratio as a point Z divides the line segment AB.

8. Let $ABCD$ be strictly convex quadrilateral and let the points $T_a,T_b,\ T_c,T_d$ be centroids of the triangles BCD,ACD,BAD,ABC, respectively. Prove that the medians of the quadrilaterals $ABCD$ and $T_aT_bT_cT_d$ are concurrent.

9. Given an equilateral triangle ΔABC let a be an affix of the vertex A. Determine the affix of the vertex B if the origin coincides with:

a) the vertex C,

b) the centroid T of the ΔABC,

c) A_1, the foot of the altitude at the vertex A to the line segment BC.

10. If a,b and c are complex numbers so that they satisfy

$$a^2+b^2+c^2=ab+bc+ca,$$

then either $a=b=c$ or a,b and c are affixes of the vertices of an equilateral triangle. Prove it!

11. Given a triangle $A_1A_2A_3$ and a point P_0 on its plane. Let $A_s=A_{s-3},\quad s\geq 4$. We construct consecutive points $P_1,P_2,P_3,\ldots$ such that the point P_k, under rotation around the point A_{k+1} at $-\frac{2\pi}{3}$, maps at P_{k+1}. If $P_{2013}=P_0$, then the triangle $A_1A_2A_3$ is an equilateral triangle. Prove it!

12. The points $a=1+i$ and $c=-1+i\sqrt{3}$ are opposite vertices of a square. Determine the other two vertices of that square.

13. Given a square $ABCD$ and a as the affix of A. Determine the affixes b,c,d of B,C,D if the origin coincides with:

a) the vertex B,

b) the vertex C,

c) the centre of the square.

14. Squares are constructed to the outside of the quadrilateral $ABCD$ on each side. If A', B', C' и D' are centres of the squares constructed on the sides AB, BC, CD and DA, respectively, E is the midpoint of $A'C'$, F is the midpoint of BD, G is the midpoint of $B'D'$ and H is the midpoint of AC, prove that the quadrilateral $EFGH$ is a square.

15. Let a,b be complex, but not real numbers, and are such that they satisfy the following $|a-b|=2$ and $ab=1$. Prove that the quadrilateral $ABCD$ whose vertices have affixes $-1,a,1,b$, respectively, is an isosceles trapezoid.

16. Let a,b,c be complex numbers such that they satisfy the following

$$a+b+c=0 \text{ and } |a|=|b|=|c|.$$

Then a,b,c are vertices of an equilateral triangle. Prove it!

17. The squares $BCDE, CAFG$ and $ABHI$ are constructed to the outside of the triangle ΔABC, on each side BC,CA and AB. Let $GCDQ$ and $EBHP$ be parallelograms. Prove that the ΔAPQ is an isosceles right-angled triangle.

18. Let $ABCD$ be a convex quadrilateral so that $\overline{AC}=\overline{BD}$. On the exterior side of the quadrilateral on its sides are constructed equilateral triangles. Let O_1,O_2,O_3,O_4 be the centres of triangles constructed on the sides AB,BC,CD,DA, respectively. Prove that the lines O_1O_3 and O_2O_4 are perpendicular to each other.

19. The quadrilateral $ABCD$ is inscribed into a circle, such that AC is its diameter. Lines AB and CD meet at M, and the tangents at B and D meet at N. Prove that $MN \perp AC$.

20. Let $ABCD$ be a cyclic quadrilateral. The points P and Q are symmetric to C concerning the lines AB and AD, respectively. Prove that PQ passes through the orthocentre of the triangle ABD.

21. Let ABC be a triangle so that the tangent of its circumcircle at the vertex A meets the midsegment of a triangle (parallel to BC) at A_1. The points B_1 and C_1 are defined analogously. Prove that the points A_1, B_1, and C_1 are collinear and the line which passes through these points is perpendicular to the Euler line of ΔABC.

22. Let $ABCD$ be a cyclic quadrilateral. Prove that

a) the Euler circles of the triangles ABC, BCD, CDA, DAB meet at a unique point.

b) the centres of the Euler circles of the triangles ABC, BCD, CDA, DAB are vertices of a cyclic quadrilateral.

23. Prove that the Symson line of any point P (P is a point placed on a circumcircle of ΔABC) bisects a line segment PH where H is the orthocentre of a ΔABC.

24. Let ΔABC be such a triangle that $\overline{AB} \neq \overline{AC}$ and let D be a point of intersection of the tangent to the circumcircle of ΔABC at A and the line BC. If E and F are such points of bisectors of line segments AB and AC, respectively, that the lines BE and CF are perpendicular to BC, then the points D, E and F are collinear. Prove it!

25. (Brokar theorem). Let $ABCD$ be a cyclic quadrilateral. The lines AB and CD intersect at E, the lines AD and BC intersect at F and the lines AC and BD intersect at G. Prove that the circumcentre O of the quadrilateral coincides with the orthocentre of ΔEFG.

26. Let $ABCD$ be a cyclic quadrilateral and let K, L, M, N be the midpoints of the sides AB, BC, CD, DA, respectively. Prove that the orthocentres of the triangles AKN, BKL, CLM, DMN form a parallelogram.

27. A circle centred at O is the incircle of a quadrilateral $ABCD$ and tangents the sides AB, BC, CD, DA at K, L, M, N respectively. The lines KL and MN meet at S. Prove that $OS \perp BD$.

28. (Newton theorem). Let $ABCD$ be a cyclic quadrilateral. Let M and N be the midpoints of the diagonals AC and BD and S be the centre of its incircle. Prove that M, N and S are collinear.

29. The incircle of a triangle ABC tangents the sides BC, CA, AB at D, E, F, respectively, and X, Y, Z are the midpoints of the sides EF, FD, DE, respectively. Prove that the centre of the incircle is placed on the line determined by the circumcentres of the triangles XYZ and ABC.

30. Let ΔABC be any triangle, with orthocentre H, circumcentre O, incentre I and K the point where the side BC tangents the incircle of ΔABC. If $IO \parallel BC$, then $AO \parallel HK$. Prove it!

31. Let O and R be the circumcentre and the circumradius of ΔABC, and Z and r be the incentre and the inradius of ΔABC, respectively. If K is the centroid of the triangle whose vertices are the points where the incircle tangents the sides of ΔABC prove that $Z \in OK$ and also that $\overline{OZ} : \overline{ZK} = \frac{3R}{r}$.

32. Let F be the point on the base AB of a trapezoid $ABCD$, such that $\overline{DF} = \overline{CF}$, $E = AC \cap BD$ and O_1 and O_2 be the circumcentres of the triangles ADF and FBC, respectively. Prove that $FE \perp O_1O_2$.

33. Let the tangents of a circle Γ at points A and B meet at C. The circle Γ_1 is such a circle that passes through C, tangents the line AB at B, and meets Γ at M. Prove that the line AM bisects the line segment BC.

34. Let H be the orthocentre of a ΔABC. The tangents at A to the circle whose diameter is BC touch the circle at P and Q. Prove that the points P, Q and H are collinear.

35. Prove that the area of the triangle whose vertices are feet of the perpendiculars at any vertex of a cyclic pentagon to its sides does not depend on the choice of the vertex of the pentagon.

36. The feet of the altitudes at the vertices A,B and C of a ΔABC are D,E and F, respectively. The line through D is parallel to EF and meets the lines AC and AB at Q and R, respectively. The line EF meets the line BC at P. Prove that the circumcircle of ΔPQR consists of the midpoint of the side BC.

37. Given a square $ABCD$ and a circle Γ with a diameter AB. Let P be any point on the side CD, M and N be the points where the line segments AP and BP meet Γ which differs from A and B and Q be the intersection of the lines DM and CN. Prove that $Q\in\Gamma$ and further that $\overline{AQ}:\overline{BQ}=\overline{DP}:\overline{CP}$.

38. Given a right-angled triangle ΔABC whose right angle is at B and sides $\overline{AB}=4,\ \overline{BC}=3$. A point E is the midpoint of the side AB, and the point D is on placed the side AC and moreover $\overline{DA}=1$. Let $F=DE\cap BC$. Determine the length of the line segment BF.

39. Let $A_0A_1...A_{13}A_{14}$ be a regular 15-gon. Prove that $\frac{1}{A_0A_1}=\frac{1}{A_0A_2}+\frac{1}{A_0A_4}+\frac{1}{A_0A_7}$ holds.

40. Given a triangle ABC and points P,N,M positioned on the sides AB,BC,CA, respectively. Prove that the circumcircles of the triangles APN,BMP,CNM meet at a unique point.

41. In a convex quadrilateral $ABCD$ the sides AB and CD are congruent.

a) The lines AB and CD with the line which connect the midpoints of the sides AD and BC form congruent angles. Prove it!

b) The lines AB and CD with the line which connect the midpoints of the diagonals AC and BD form congruent angles. Prove it!

42. In a convex quadrilateral $ABCD$ the points P and Q are the midpoints of the diagonals AC and BD, respectively. Prove that

$$\overline{AB}^2+\overline{BC}^2+\overline{CD}^2+\overline{DA}^2=\overline{AC}^2+\overline{BD}^2+4\overline{PQ}^2.$$

43. Let ABC be an acute scalene triangle so that $\overline{AC}>\overline{BC}$, O be the circumcentre H be the orthocentre, and F be the foot of the altitude at the vertex C. Let P be a point on the line AB, which differs from A, so that $\overline{AF}=\overline{PF}$, and M be the midpoint of the line segment AC. Let X be the point of intersection of the lines PH and BC, Y be the point of intersection of the lines OM and FX, and Z be the point of intersection of the lines OF and AC. Prove that the points F,M,Y and Z lie on the same circle.

4. Conclusion

Continuous work with mathematically gifted students is a basic prerequisite for their faster progress. It should take place according to a curriculum appropriate for the age of the students, which we have previously presented for students aged 17–18. Among other things, we believe that the acquisition of theoretical knowledge provided by this curriculum, supported by the collections of problems will allow:

- Improving the qualities of students' thinking,
- Students to apply the types of inferences correctly: induction, deduction, and analogy,

- Students apply the scientific methods correctly: observation, comparison, experiment, analysis, synthesis, classification, systematisation and the axiomatic method,
- Students to acquire the necessary knowledge for their future development.

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