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Students' conception of partial derivatives of a function of several variables

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Abstract

The study aimed to explore and assess students' conceptual understanding of partial derivatives in functions of several variables. A case study research design was employed, with four classes randomly selected for participation. Qualitative data were collected through interviews and open-ended questions developed by the researchers. The analysis was conducted using APOS theory, based on the proposed genetic decomposition. The findings reveal that the majority of students' understanding falls under action conception, interpreting partial derivatives similarly to single-variable derivatives. Additionally, many students struggled to extend their knowledge from single-variable derivatives to partial derivatives and faced difficulties in converting between algebraic/symbolic and graphical representations. Based on these results, the researchers recommend that educators adopt instructional strategies that foster deeper conceptual understanding, rather than relying on traditional teaching methods.

Keywords: APOS Theory; conceptual understanding; function of several variables; partial derivatives

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1. INTRODUCTION

The importance of learning mathematics for engineering and technology students is to provide them with the skill of working with several mathematical ideas and various representations in different areas of their day-to-day activities (Athavale et al., 2021). However, different literatures indicate that mathematics in general, and Applied Mathematics, in particular, is one of the most difficult fields of study for undergraduate engineering and technology students (Cornu, 1981; Eisenberg, 1991; Kashefi et al., 2010, Schwarzenberger, 1980; Tall, 1992; Forde et al., 2023).

The origin of the function concept is rooted in the study of natural phenomena as a means of describing, explaining, and predicting them (Cho, 2013). The modern type of function concept goes back to the innovation of analytic geometry that stimulated the notion of function at the time of Descartes and Fermat in the 17th century (Atkinson, 2002; Borji et al., 2024). Understanding the concept of a function is a prerequisite to understanding mathematical concepts in general and functions of several variables in particular. Students will find it challenging to understand concepts of functions of several variables if they are having trouble with concepts of functions of a single variable (Schwarzenberger, 1980; Tall, 1992; Kurysheva et al., 2023).

Literature indicates students have problems understanding different concepts of calculus like derivatives of a function and their application. According to Desfitri (2016), the reason for the lack of conceptual understanding of concepts of calculus is problem-related to mastering the prior concept. It is so obvious that if the students lack an understanding of its pre-requisite concepts, then they will have difficulties understanding a concept (Diekei and Isleten, 2004; Kultur et al., 2011). Bingolbali (2008) asserts that prior knowledge of functions, limits, continuity, slope, and rate of change is necessary for students to comprehend derivatives. There are various ways to describe the idea of a derivative, including graph, symbolic, and numerical representations (Desfitri, 2016).

1.1. Conceptual background

Mohammed (2014) states that the foundation of about 75% of engineering courses is mathematics. Without mathematics, it is impossible to discuss science, engineering, and technology (Winkelman, 2009). Given that mathematics is seen as the foundation of engineering and that its study aims to comprehend the patterns that permeate both the universe and the mind (Schoelfeld, 1992), it is clear that handling mathematical learning with care and attention is crucial. The absence of conceptual comprehension among engineering and technology students, on the other hand, has been reported by numerous researchers (Goold, 2012; Huang, 2010; Mohammed, 2014).

One of the main causes is that students struggle to understand mathematical topics because they lack conceptual understanding (Handhika et al., 2016; Borji et al., 2023). Research shows that various ideas in the calculus of single and many variables were not conceptually understood by pupils (Martin-Blas et al., 2010; Martinez-Planell & Trigueros Gaisman 2009). The purpose of this study was to gauge the conceptual knowledge of students regarding the relationship between several factors. To confirm students' conceptual comprehension based on the APOS theory, the definition which includes expanding the definition to new concepts and representations was taken into consideration. These representations might be graphical, algebraic, or symbolical.

1.1.1. Genetic decomposition

The idea of partial derivatives of a function of many variables was categorized by the researchers into three categories: definition, extension to a new concept, and algebraic/symbolic representation and graphical representation. To more accurately depict what students do, the researchers piloted and improved the initial genetic decomposition they had presented. The results can be found below:

1. Students were asked to determine extreme values of functions of a single variable and they were asked to rotate it about the x-axis and the y-axis. They were also asked to move this concept to a function of several variables.

2. Students were asked to find derivatives for a single variable. They were also asked to move this concept of functions of several variables.

The refined genetic decomposition was proposed after the pilot study.

1. The definition, extension, and statement of partial derivative rules and principles, as well as their application to a function of two variables, are made possible by an action conception. These rules are provided in algebraic/symbolic form.

2. Determining the partial derivative and applying it to functions involving several variables is made possible by a process conception. This could entail looking at the function's structure to see if a rule can be applied or if the function should be constructed in a way that makes it possible to apply the right rules to solve a particular problem.

3. Seeing a series of operations as a whole and executing written or mental operations on the internal organization of the provided functions of many variables are made possible by an object conception. This allows one to turn algebraic representations into graphical representations and vice versa.

4. Putting the idea of a function of two variables into order and connecting the action, process, and object into a logical framework. The functions of multiple variables are interpreted differently in different contexts, and there are several ways to find the extreme values of these variables. These include: [a] finding the partial derivative and applying it to functions of multiple variables; [b] applying various properties and rules in finding the partial derivative and applying it to functions of multiple variables; [c] setting up a partial derivative to represent gradient and directional derivatives of functions of multiple variables; and [d] finding the partial derivative.

1.2. Purpose of study

The objective of this study is to examine students' conceptual understanding of partial derivatives of a function of several variables.

The study's goals are as follows:

- To determine the level of students' conceptual understanding of concepts of partial derivatives of a function of several variables.
- To investigate which method of instruction contributes more to students' conceptual understanding.

The research questions are as follows:

- What is the level of student's conceptual understanding of concepts of partial derivatives of a function of several variables?
- Which method of instruction contributes more to students' conceptual understanding?

2. METHOD AND MATERIALS

2.1. Research design

The qualitative research approach emphasizes the importance of bringing word or picture data for a detailed description of the phenomenon to better comprehend students' concepts of partial derivatives of a function of several variables. Consequently, a case study research method was used to investigate how well students understood the function of several variables conceptually.

2.2. Participants

The present investigation involves the random selection of three departments, namely Mechanical Engineering, Garment Engineering, and Textile Engineering, with all students in each department participating in the study. For interviews, three students were specifically chosen from each department.

2.3. Data collection tools

The researchers created two-tiered conceptual examinations and interview questions to gather sufficient data to investigate students' understanding of partial derivatives of a function of many variables.

2.4. Data analysis

The APOS theory framework was used to examine the data, which were divided into four categories: definition, extending definition, algebraic/symbolic representation, and graphical representation of partial derivatives of a function of two variables.

3. RESULTS

The concepts of continuity and differentiability of functions at a given point were not new to the students. They already discovered that all differentiable functions are continuous, even when the opposite is not true, in a function of a single variable. The most widely used example to concisely explain this idea is f (x)=|x| at x=0. Since f is defined at x=0 by lim┬(x→0)⁡〖|f(x)〗 =f(0)=0, it follows immediately that f is continuous at x=0. That being said, since f^' (0^-)≠f^' (0^+), f has no derivative at x=0. Consequently, at x=0, f (x) is not differentiable. Students are expected to be able to apply this past knowledge to the novel idea of a function with several variables. Additionally, it was anticipated that the students would apply their knowledge of single-variable derivatives to partial derivatives of two-variable functions.

The derivative of a function of a single variable, f(x), is defined as the slope of a tangent line at a certain location and the rate of change concerning changes in the variable x. Students struggled to apply this idea to derivatives of functions involving multiple variables, though. Most students were able to find the partial derivatives of f(x,y) concerning x and Gasman y while maintaining one of the variables constants. However, they struggled to interpret gradients, directional derivatives, the function's differentiability, extreme values of the function, and multiple representations of the function.

Because of this, students had to select the right response and provide evidence for it based on several theories about partial derivatives of a function of two variables. Their answers were divided into various mental constructs following the APOS theory, which is based on genetic decomposition and is proposed based on four themes: defining concepts of the derivative of a function of a single variable to a function of multiple variables, algebraic representation/symbolic representation, and graphical representation of a particular concept. This helps the researcher to determine how well the students comprehend partial derivatives and how to apply them.

In a function of a single variable $f(x)$, a derivative of a function concerning x is given by $f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$. In this definition $f(x+h)- f(x)$is the change in the output that results from adding h to the input. Similarly, in defining the partial derivative if $f(x,y)$ is a function of two variables, then the partial derivative of the function concerning x is given by $f_x(x,y) = \lim_{h\to 0}\frac{f(x+h,y)- f(x,y)}{h}$, whereas the partial derivative of the function concerning y is given by $f_y(x,y) = \lim_{h\to 0}\frac{f(x,y+h)- f(x,y)}{h}$ (Adams, 2003; Stewart, 2008).

With this idea in mind, the students were given a function f and g such that f(x,y)=g(x,y)+k, where k is a constant. They were then asked to apply the property of the derivative of the constant function concept and determine a partial derivative of f concerning the variable x following the definition of partial derivative concepts. The information shows that, in general, and in particular, the respondents did not fully comprehend the derivative of a constant function and the derivative of a constant term presented in a function of two variables. The information verifies that a constant function k's derivative is:

- *Derivative of a constant function is a function itself i.e.* k
- *The derivative of a constant function is one i.e. (*$k' = 1$*)*
- *The derivative of a constant function is zero*

As indicated above, few students considered the derivative of a constant term within the given function of two variables as it is and symbolically represented it as "$f_x(x,y) = \lim_{h\to 0}\frac{f(x+h,y)-f(x,y)}{h} = g_x(x,y) + k$" and justified it as if $f(x,y) = g(x,y) + k$, then $f_x(x,y) = g_x(x,y) + k$. They considered it a matter of adding a constant number k on the partial derivative of the given function concerning variable x because they considered that the initial function $f(x,y)$ is equivalent to $g(x,y) + k$ which is a wrong conclusion. Moreover, the respondents could not be able to apply the addition rule of the derivative that they learned in a function of a single variable and unable to extend it to the partial derivative of a function of two variables. This indicates that students have a poor prior conception of the derivative of a constant function and partial derivation of a function of two variables.

There were also a few respondents who replied the partial derivative of the given function of two variables was"$f_x(x,y) = g_x(x,y) + 1$" and they tried to justify their answer as if the derivative of a constant function is 1. This indicates that they had difficulties understanding the properties of derivatives in Applied Mathematics I and were unable to extend them to a function of two variables.

Similarly, some students replied as "$f_x(x,y) = g_y(x,y)$" and justify their answer as since $f(x,y) = g(x,y) + k$, then it is equal to $f_x(x,y) = g_y(x,y)$. Out of them, few respondents justified it as "*if* $f(x)$ *has relation with* $g(x)$*, then they must have the same corresponding value of* x *and* y *or domain and range*" which shows that the students were not able to understand the given question and what partial derivatives of the given function concerning variable x properly. According to Duval (2006), students who struggled to define and handle the provided concepts were classified as being at the conceptualization stage. According to the suggested genetic breakdown, they can therefore be grouped under action conception.

"f_x (x,y)=g_x (x,y)" was selected as the right response from the provided choices by the majority of respondents across all groups. The assertion made by several respondents that the "derivative of a constant function is zero" supports their response as valid. Students can find the partial derivative of the given function here, support their conclusions with reasoning, and accurately describe their conclusions symbolically. According to Schwarzenegger (1980) and Tall (1992), students may find it challenging to apply their existing knowledge to a new notion if they have trouble grasping the required concepts. Furthermore, according to Dubinsky & Harel (1992), Eisenberg & Dreyfus (1994), and Metcalf (2007), children who struggled to represent concepts in a variety of ways but applied concepts and rules without memorization may be at the process level. They fall within the category of process conceptualization as a result.

Regarding extending prior knowledge of the continuity of a function of two variables to a partial derivative of a function of two variables, students were also probed to differentiate whether "every differentiable function is continuous or vice versa". The data shows that the majority of the students stated that "*all continuous functions are differentiable*". Some students described that "*all differentiable functions are not necessarily continuous*" whereas few respondents described it as "$f(x)$ *is continuous means* $f'(x)$ *exist and* $f'(x) = f(x)$" which could be considered as wrong conception. This suggests that most students struggle to comprehend the connection between the function's provided function continuity and differentiability. As a result, pupils were showing signs of confusion when it came to comprehending the prosperities and theorems of a function of two variables, and they were unable to apply it to the continuity of the function. According to

Schwarzenegger (1980) and Tall (1992), students may find it challenging to apply their existing knowledge to a new notion if they have trouble grasping the required concepts.

The data reveals that some students from each group under investigation replied that "*every differentiable function is continuous functions*" which is a correct conception but even though they were correctly answered still majority of them still failed to justify their answers. Only a few of the respondents tried to justify using the concept of a function of a single variable rather than extending the concept to a function of two variables. For instance, there was a student who justified it using a function $\mathrm{f(x)} = \sqrt{x}$ Which is continuous at $[0, \infty)$ but $(\sqrt{x})' = \frac{1}{2\sqrt{x}}$Not continuous at $[0, \infty)$ whereas, few of them tried to justify it using the function of a single variable i.e. $f(x) = |x|$ Which is continuous but not differentiable at a point $x = 0$. This demonstrates that rather than expanding the idea to a function of two variables, students were applying their previous concepts of a function of a single variable. Amatangelo (2013) states that when pupils are stuck with their old schema, they develop a false perception. Students thus struggle to relate differentiability and continuity in a function of two variables. This difficulty could arise from their continued use of the concept of a function of a single variable or from their inability to extend it to a function of two variables. Following the definition of differentiability, students were also questioned about "how to determine the differentiability of a function of two variables". A function is only considered differentiable in the context of a single variable if it is continuous; nevertheless, continuity is not a prerequisite for differentiability. Differentiability necessitates that a function's graph always has a single tangent vector, while continuity demands that a function's graph be a continuous curve. Thus, if and only if the graph of a function of a single variable is a continuous, smooth curve without any sharp corners, then the function is differentiable. In a similar vein, if the graph of a function of two variables shows no folds or sharp peaks, it is differentiable. At a certain point, this surface is connected to the only tangent plane (Adams, 2003; Stewart, 2008). Nevertheless, a function of two variables is not only differentiable if a partial derivative exists. One can say that a function $f: \mathbb{R}^2 \to \mathbb{R}$ of two variables x and y is differentiable at (a, b) if

1. Both f_x and f_y Exist at the point (a, b)
2. $\lim_{(x,y)\to(a,b)} \frac{f(x,y)-f(a,b)- f_x(a,b)(x-a)- f_y(a,b)(y-b)}{|(x-a)^2+(y-b)^2|} = 0$

Having this concept, students were asked to determine whether the existence of all partial derivatives of f at a given point implies the differentiability of a function of two variables or not. Here, the majority of the students replied "*If the first partial derivative of f exists at P_o, it is necessarily differentiable*" and "the *existence of the first partial derivative of f at P_0 ensures differentiability of f at* P_0. The respondents considered the existence of a partial derivative as a necessary and sufficient condition of the differentiability of the given function at the given point. They rejected the continuity of the function and the nature of the graph of the function at the given point. This indicated that students had difficulty extending the concept of limit, continuity, and the nature of the graph of the function to answer the question in their hand. So, students' conceptions of the differentiability of the given function of two variables were weak. According to Martinez-Planell & Trigueros Gaisman (2009), pupils fall into the action conception category if they are unable to apply their existing knowledge to the new notion. The study's findings are also consistent with research by Davis (2007), Akkus et al. (2008), and Carlson et al. (2010), which found that students struggled to make connections between prior concepts and the new information represented by function representations such as equations, graphs, tables, and word forms.

Moreover, few students replied "*If f is differentiable at P_o, then it does not mean that the first partial derivative exists at P_o*" and "*if all partial derivatives of f at a point P_o Exists, then f is not differentiable at a point P_o*". While the second example demonstrates the opposite, the first discloses that a function of two variables could be differentiable even though its partial derivatives are. Both might be viewed as incorrect conceptions. This indicates that the notion of the differentiability of a function of two variables was proving to be challenging for the pupils.

Some students justified their answer as the "*existence of the first partial derivative of f at P_o does not necessarily ensure differentiability of f at P_o*", "*if all partial derivatives of f at a point P_o Exists, then f is not necessarily differentiable at P_o*" and "*if f is differentiable at P_o, then the first partial derivative exists at P_o*". All excerpts given above correctly stated the relation between differentiability and partial derivative of the given function of two variables. However, they tried to justify their answer using a polynomial function. For instance, there was a student who justified his answer using a function, $f(x, y) = 3x^2 + 3xy + 6y + 6$ is continuous and also it is differentiable. The given function is indeed both continuous and differentiable. Therefore, following the suggested genetic decomposition, they might be classified as process conception.

The derivative was a tool used to determine the maximum and minimum values of a function in a function of a single variable. Even if the circumstances are a little more complicated, partial derivatives in a function of two variables serve a similar purpose to the derivative in a function of a single variable. For example, students utilized a second derivative to evaluate the extremes of a function in a function of a single variable. For a function with two variables, there is a counterpart to the second derivative test. In other words, consider *f* to be a function of two variables and (a,b) to be an interior point within *f*'s domain. Suppose that the second partial derivatives of f are continuous on a disk centered at (a, b) and let $= f_{xx} \cdot f_{yy} - f^2{}_{xy}$.

1. If $D > 0$ and $f_{xx}(B) < 0$, then $f(a, b)$ is a local maximum
2. If $D > 0$ and $f_{xx}(B) > 0$, then $f(a, b)$ is a local minimum
3. If $D < 0$, then $f(a, b)$ is not a local maximum or minimum (Adams, 2003; Stewart, 2008)

Based on this, students were provided the graph of a function $f(x, y)$, has a maximum value at point B, and then were probed to match the graphical representation of the given function with its algebraic representation. In this question, students were already notified that point B is a maximum point and they were asked to algebraically as per the second partial derivative test they learned. The majority of the students replied with the algebraic representation of a maximum value at a given point B as "$D > 0$ and $f_{xx}(B) > 0$" and they attempted to defend their answers by arguing that since the second partial derivative of the given function concerning variable x is larger than zero, students should have believed that the second partial derivative of the given function at point B is maximal when it is positive. Here, a positive value is associated with a maximum value of the function which is a wrong conception. According to Monk (1992) and Oehrtman et al., (2008), students' difficulty in interpreting graphs of a given function could be considered as a lack of understanding of the concept.

Some students who respond as "$D < 0$ and either $f_{xx}(B) > 0$ or $f_{xx}(B) < 0$" and few students replied as "$D < 0$ and $f_{xx}(B) < 0$". This shows that the students were not clear about the properties of extreme values of functions of several variables or were unable to understand the given graph properly. This suggests that they encountered challenges when attempting to translate a function of two variables from a graphical to an algebraic representation. Research from a variety of sources, including Davis (2007), Akkus et al. (2008), Drlik (2015), Oehrtman et al. (2008), and Monk (1992), has shown that students struggle greatly to connect knowledge presented in equation form and across different representations. As a result, according to these findings, the majority of pupils fall into the lower conceptual level (Duval, 2006).

Derivate was defined as the instantaneous rate of change of a function as the variable changes and the slope of a tangent line in a function of a single variable. In Stewart (2008) partial derivatives are interpreted as *rates of change*. If $z = f(x, y)$, then f_x Represents the rate of change concerning x when y is fixed. Similarly, f_y Represents the rate of change concerning y when x is fixed. Furthermore, the instantaneous rate of change of a function concerning any one of its independent variables was understood by several mathematical reference books as the partial derivatives of a function of two variables, and as a tangent plane to the level surface. The gradient of a function *f(x,y),* which is represented as the sum of a vector form of its first partial derivatives, is thus included in the concept of this partial derivative. With those ideas in mind, the data shows that students defined the gradient of a function *f(x,y)* as follows before receiving any intervention:

- a vector form of its first partial derivatives
- as the instantaneous rate of change of the $f(x, y)$ with respect to any one of its independent variables
- a tangent plane to the level surface
- the instantaneous rate of change of the function f concerning any one of its independent variables, used to describe a tangent plane to the level surface and a vector form of its first partial derivatives

All of the respondents listed above correctly describe what a gradient of the function is even though the deepness of their responses varies. Despite that students used the same textbook i.e. Stewart (2008) they came up with different conceptions. Those conceptions might have emerged either due to their prior conception or from different reference books they used in the library. For example, most students defined a function's gradient as the vector form of its first partial derivatives, but some also defined it as the function's rate of change concerning any one of its independent variables and as the tangent plane to the level surface, as shown above.

Furthermore, relatively few students understood gradient to mean rates of change, a vector form of partial derivatives of the provided function of two variables, or a tangent plane to the level surface. The accuracy of the definition given by those respondents was very high. They extended the concept of the gradient to other different concepts like rates of change, tangent plane, and parietal derivatives. Thus, they could be categorized under process conception.

Besides, a few students defined it as "*the level of change of a function*", "*a function that we get from a normal equation"* and "*a slope and slope mean first derivative"*. These definitions were not as clear in line with the definition of gradient and were not elaborated verbally, algebraically, symbolically, or graphically. Thus, it can be considered a misconception. For instance, students defined gradient as a slope whereas slope means a first derivative of the given function. The slope could indeed be interpreted as the first derivatives in a function of a single variable but not lack extending the concepts to a function of two variables. As a result, pupils found it challenging to relate the idea of slope represented by the first derivative of a single variable to the idea of gradient represented by the partial derivative of a function of two variables of a function of two variables.

There are also a few students who defined gradient as "*a vector form of its second partial derivatives*". It is incorrect to think of the defined gradient as a second partial derivative because a function of two variables comprises two distinct variables. They would therefore fall within the category of action conceptualization.

With a few exceptions that might be classified as having a process conception, the study shows that most students' conceptions of the partial derivative of a function of two variables were inadequate and did not surpass action conception. Not a single student attempted to use a graphical depiction to support their answers. The literature demonstrates that there are various ways to portray concepts of derivatives, including graphical, symbolic, and numerical representations (Desfitri, 2016; Stewart, 2008). In contrast to this, none of the students were able to express the given concept using different representations. Such students' difficulty could be due to a lack of understanding of prior concepts like slope, limit, continuity, derivative, and their application in different concepts. A study conducted by Desfitri (2016) reveals that more than 45% of students were challenged to understand the concept of derivative due to less mastering the prior concept. Similarly, studies conducted by

Diekei & Isleten (2004) and Kultur et al., (2011) show that if the students had a lack of understanding of its pre-requisite concepts, then they will have difficulties in understanding a new concept. Additionally, according to Bingolbali (2008), students must grasp earlier ideas like function, limit, slope, continuity, and rate of change to comprehend derivatives. As a result, the study's findings are rather similar to the previously mentioned research findings. According to the study, students found it challenging to define, comprehend, expand, and illustrate various ideas related to a function of two variables in general and partial derivatives in particular.

Following the implementation of the intervention, the researcher noted that certain students persisted in incorrectly defining, interpreting, and representing a function of two variables, either in the same manner or in a different one. Students in Applied Mathematics I were taught that the derivative of a function of a single variable indicates the function's rate of change as x changes. This understanding of the derivative of a single variable function is significant. However, in Applied Mathematics II the concept was extended to several variables. It is expected from the students to understand and justify that if the variables in the given function are more than one and they fix one variable and let the other change, or if they allow more than one variable to change what will happen? To answer such questions, students should be familiar with the concept of partial derivatives.

So, students were provided the function $f(x, y) = \frac{2xy}{x^2+y^2}$ And asked to define partial derivatives, determine partial derivatives at a point$(0,0)$, which rule of derivatives they used, determine the differentiability of the given function at a point$(0,0)$, and the association between continuity and differentiability of the given function at a point $(0,0)$ through interview.

It is obvious that if students want to find the partial derivative of $f(x, y)$ Concerning x, they should hold y as a fixed (constant) and allow x to vary, then students are looking for the partial derivative of f concerning x which can be denoted by $f_x(x, y)$. Similarly, if they hold x fixed and allow y to vary, then students are looking for the partial derivative of f concerning y which can be denoted by $f_y(x, y)$ (Adams, 2003; Stewart, 2008). Besides, the interviewer asked the respondents which properties of derivative they were using and students knew that a derivative of any constant function is zero in a function of a single variable. This holds in the case of partial derivatives of functions of several variables.

All interviewees have no difficulty finding partial derivatives of a given function. They applied a quotient rule to get partial derivatives. f_x and f_y, holding y and x as a constant respectively which is the correct conception. Applying the laws of derivatives and expanding the idea of derivatives from a function of one variable to a function of two variables was not a challenge for the responders. They can determine partial derivatives of a function based on property derivatives for unrestricted domains.

Moreover, respondents can apply different rules of derivatives (like for instance, a derivative of a constant function, product rule, quotient rule, etc.) to determine partial derivatives of an unrestricted domain. This shows that they have interiorized the actions described. Students at this level should be able to identify which derivative properties to use to find the partial derivatives of a given function involving two variables.

In addition, students can expand the idea to find the differentiability of a function at a specific point and calculate the partial derivative of some limitations. There is a fundamental distinction between process and action conceptions. If this is the case, students proved to have internalized the process of determining partial derivatives for the specified function involving two variables. Therefore, they fall under the same category as process conceptualization, as suggested by genetic decomposition. For instance, respondents were asked to determine partial derivatives of the given function $f(x, y) = \frac{2xy}{x^2+y^2}$ At $(0,0)$. None of them answered correctly. They replied it is the partial derivative of $f(x, y)$ At $(0,0)$ does not exist.

On the other hand, they could not able to reason out what partial derivative stands for. This shows that they have the problem of conversion. They had difficulty relating the first partial derivatives. $f_x(x,y)$ and $f_y(x,y)$ of a function of two variables at a given point (x,y) to rates of change of $f(x,y)$ measured in the direction of the positive x −and y −axis, respectively. If we want to know how fast $f(x,y)$ changed as we move through the domain of f at (x,y) in some other direction, it requires concepts of directional derivative.

Moreover, students were asked to extend the concept of limit to continuity, then continuity to partial derivatives, and partial derivatives to differentiability a function of two variables at a point $(0,0)$ on the given function above. For instance, some of them answered as "$f(x,y)$ *is not differentiable at* $(0,0)$ *because the limit of the function doesn't exist*". Besides, the researcher asked the respondent whether the given function is continuous or not at a given point$(0,0)$. His response was "*not continuous because the limit of the function doesn't exist and the function is not differentiable at* $(0,0)$".

Derivative is used to determine extreme values, monotonicity, and concavity of a function as a function of a single variable. Similarly partial derivative helps us to determine extreme values and rate of change of functions of several variables. So, students were probed to convert word problems to algebraic representation, algebraic representation to graphical representation, and graphical representation to algebraic representations. If they can coordinate to identify extreme values of function, and rate of changes of functions within a given domain of the function using concepts of partial derivatives, and as indicated by the suggested genetic decomposition, they can then be categorized under the object conception.

Students were probed through the question "Let temperature at any point (x,y) of the xy plane is given by $T(x,y) = y^2e^{-x}$. In what direction at the point $(-1,1)$ does the temperature increase more rapidly? Find the maximum rate of change?"

The students know how to find the partial derivatives of a given function of two variables. The partial derivatives f_x, and f_y show the rate of change of a function f when we change x while keeping y constant and, correspondingly, when we change y while keeping x stable (Adams, 2003; Stewart, 2008). Students were probed to extend concepts of partial derivatives to gradient vector and directional derivatives as below.

Gradient is a vector form of the first partial derivative of a function$f(x,y)$. This vector is important to determine the rates of change of a function$f(x,y)$ called directional derivative when we allow both x and y to change (Adams, 2003; Stewart, 2008). So, the students were probed on how to determine the rate of change of $f(x,y)$ if we allow both x and y to change and how to determine the maximum values. For instance, all respondents understood that the given equation of temperature as a function $T(x,y) = y^2e^{-x}$ And the given function is a function of two variables.

To determine the direction at which the temperature increases most rapidly, the respondents need to know the concept of gradient and find the gradient vector at the given point. The data reveals that respondents were able to find the partial derivative of the given function $T(x,y) = y^2e^{-x}$ as "$T_x(x,y) = -y^2e^{-x} \Longrightarrow T_x(-1,1) = -e$ and $T_y(x,y) = 2ye^{-x} \Longrightarrow T_y(-1,1) = 2e$". Then, the gradient vector becomes $\nabla T(x,y) = -ei + 2ej$ which is correct. But only a few students answered that "*the temperature most rapidly increases in the direction of gradient vector*" whereas the maximum value of the temperature could be determined as "*a magnitude of gradient*". Thus, they could be categorized under process conception. This suggests that the respondents had a thorough understanding of the definition of gradient, the nature of the function, and gradient vector determination. Trigueros and Martinez-Planell (2009, 2010, and 2011) state that respondents can be categorized as belonging to process conception if they can handle and convert a given function between multiple representations and do not play a significant role in the creation of an object conception.

On the other hand, the data reveals that the majority of the respondents were not able to extend the concept of the gradient to the steepest incline of the graph of the function at a point$(-1,1)$ and were unable to find the maximum value of the temperate if exists. They have difficulty differentiating gradient vectors from directional derivatives. For instance, some respondents answered the direction in which the temperature most rapidly increases as:

- *in the direction of the gradient along the y-axis*
- *in the direction of the position vector*
- *a maximum value of the function at the given point*
- *in the direction of the directional derivative*

Whereas, the maximum value of the temperature T at $(-1,1)$ is:

- if $T' > 0$ *using the first derivative test*
- *a dot product of the first partial derivative with a unit vector*
- If $D > 0$ and $T_{xx} < 0$ *using the second partial derivative test*

The data were respondents who associated the direction of the steepest incline with a gradient of the function with the y-axis, with a maximum value of the function, position vector, and directional derivatives which could be considered as a wrong conception. This can be the result of a poor comprehension of the definition and representation of the gradient vector. Carlson (1997), Dubinsky & Harel (1992), Eisenberg & Dreyfus (1994), and Metcalf (2007) state that to have a strong conceptual grasp, students must be able to discern aspects of a function from various representations and comprehend formal definitions.

Moreover, some respondents related the concept of the maximum value of the function with the first derivative test of a function of a single variable. So, they found first partial derivatives at a point $(-1,1)$ and then compared their values. The greater the value the maximum value of this function. Then this maximum value is also taken as the point at which the temperature rapidly increases. While some responders used the dot product of the first partial derivative with a unit vector, few attempted to find the maximum temperature using the second partial derivative test of a function of two variables. This suggests that pupils applied the new concepts directly to their existing knowledge. The findings support Martinez-Planell & Trigueros Gaisma's (2009) research, which demonstrates that students' comprehension is similar to that of a function of a single variable.

The students' conceptions were probed to extend the concept of gradient and unit vector to determine directional derivative in the given direction of vector provided that "Let $f(x,y) = x\sin y$. Find a directional derivative at the point $\left(3,\frac{\pi}{4}\right)$ in the direction of vector $< 1, 1 >$." All respondents applied a product rule to get the first partial derivatives concerning variables x and y respectively to get $f_x(x,y) = siny \Rightarrow f_x\left(3,\frac{\pi}{4}\right) = sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}i$ and $f_y(x,y) = xcosy \Rightarrow f_y\left(3,\frac{\pi}{4}\right) = 3cos\frac{\pi}{4} = \frac{3\sqrt{2}}{2}j$. This implies that the respondents have no difficulty determining partial derivatives of a restricted domain and applying rules of partial derivatives. Thus, they could be categorized under action conception.

Regarding directional derivative, the majority of the respondents state as "*a dot product of gradient and vector*" which is a wrong conception. According to Carlson (1997), Dubinsky & Harel (1992), Eisenberg & Dreyfus (1994), and Metcalf (2007), this suggests that the respondents struggle to describe directional derivatives algebraically and to transform a given vector into a unit vector. Other responders also transformed the provided vector into a unit vector in the direction of u, which is subsequently dot product with the gradient above. A directional derivative is the dot product of the gradient vector and the unit vector in the direction of the given vector and is represented as it "$D_u\nabla f(x,y) = \left(\frac{\sqrt{2}}{2}i + \frac{3\sqrt{2}}{2}j\right).\left(\frac{i}{\sqrt{2}} + \frac{j}{\sqrt{2}}\right) = \frac{1}{2} + \frac{3}{2} = 2$ $\frac{1}{2} + \frac{3}{2} = 2$" which is correct. This suggests that a few participants possessed a mental understanding of the concept of a partial derivative of a function and its application in determining the directional derivative of the provided function. They can therefore be categorized as part of process conception. This suggests that the respondents had a thorough understanding of the definition of gradient, the nature of the function, and gradient vector determination. It is consistent with the findings of Trigueros and Martinez-Planell (2009, 2010, and 2011), which indicate that the respondents are not significant in the process of constructing an object idea.

As we see from the pre-conceptual reasoning part, the majority of the respondents were not able to convert graphs to algebraic representation. The interview enables the researcher to understand whether the respondents can convert the graph to an algebraic representation after the intervention was given to the treatment groups. So, students were given the graph of a function $f(x, y)$ As below. They were asked to show where maximum and saddle points exist, and how to determine them algebraically.

All respondents indicated that B is a maximum value whereas a point at which the graph shifts up and down at the same time down point B is a saddle point. Even though, all of the respondents indicated where a maximum value is and saddle point could exist, only two respondents were able to represent corrected s "$D > 0$ and $f_{xx}(B) < 0$". And none of them tried to represent saddle point algebraically except one respondent who represented it as " if $f_{xx}f_{yy} - (f_{xy})^2 < 0$, then we have saddle point" which is a correct conception. Since they were able to define, transform, and depict the given graph following the suggested genetic decomposition, they can be categorized as process conception.

This suggests that the preexisting conception was still being used by the majority of responders. Algebraic representation of a maximum value at a given point B as "$D > 0$ and $f_{xx}(B) > 0$" and justify as if the maximum value of a function is greater than zero. This indicates that students' conception of maximum value is still unchanged. This finding is consistent with studies by Monk (1992) and Oehrtman et al. (2008) that found students struggled to understand function diagrams. The majority of students are therefore categorized as having lower-level concepts, according to the findings (Duval, 2006). According to the suggested genetic decomposition, none of the students who were interviewed were able to appropriately and sufficiently display all four schemata levels (action, process, object, and schema).

4. CONCLUSION

According to the study, students struggled with definitions, applying definitions to new ideas, and coming up with different methods to describe a particular function. The outcome suggests that learners encountered challenges when attempting to convert algebraic or symbolic representations into graphical representations and vice versa. Students in this study were asked questions such as algebraic expressions during the post-interview and mathematical statements throughout the reasoning section.

Students had trouble translating mathematical assertions into algebraic/symbolic and graphical representations in both situations, as well as translating algebraic representations into graphical form. According to a variety of literary works, conceptual comprehension is the ability to portray a notion in many ways. Furthermore, only a few pupils developed a process idea, and none of them reached the object level. This demonstrates that pupils have trouble deciphering the functions of multiple variables shown graphically.

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REFERENCE

Adams, R. (2003). *Calculus: A Complete Course.* Pearson Education Canada Inc., Toronto, Ontario.

Akkus, R., Hand, B., & Seymour, J. (2008). UNDERSTANDING STUDENTS'UNDERSTANDING OF FUNCTIONS. *Mathematics Teaching*, (207), 10. https://iro.uiowa.edu/esploro/outputs/magazineArticle/UNDERSTANDING-STUDENTS-UNDERSTANDING-OF-FUNCTIONS/9984374360102771

Amatangelo, M. L. (2013). *Student understanding of limit and continuity at a point: a look into four potentially problematic conceptions*. Brigham Young University. https://search.proquest.com/openview/f9019c99133c10a198de68e0c121c231/1?pq-origsite=gscholar&cbl=18750&diss=y

Athavale, P., Mondal, S., & Rivera, S. (2021). Factors influencing success in advanced engineering mathematics courses: A case study. In *Frontiers in Education, 6*, 662380. https://www.frontiersin.org/articles/10.3389/feduc.2021.662380/full

Atkinson, L. (2002). Where do functions come from? *The College Mathematics Journal*, *33*(2), 107-112. https://www.tandfonline.com/doi/pdf/10.1080/07468342.2002.11921927

Bingolbali, E. (2008). Suggestions for learning difficulties and conceptual understanding of the concepts of derivatives. *J Mathematical Misconception and Solution*, 223-253.

Borji, V., Martínez-Planell, R., & Trigueros, M. (2023). University Students' Understanding of Directional Derivative: An APOS Analysis. *International Journal of Research in Undergraduate Mathematics Education*, 1-30. https://link.springer.com/article/10.1007/s40753-023-00225-z

Borji, V., Martínez-Planell, R., & Trigueros, M. (2024). Students' geometric understanding of partial derivatives and the locally linear approach. *Educational Studies in Mathematics*, *115*(1), 69-91. https://link.springer.com/article/10.1007/s10649-023-10242-z

Carlson, M. P. (1997). Obstacles for College Algebra Students in Understanding Functions: What Do High-Performing Students Really Know? *AMATYC review*, *19*(1), 48-59. https://eric.ed.gov/?id=EJ567915

Carlson, M., Oehrtman, M., & Engelke, N. (2010). The precalculus concept assessment: A tool for assessing students' reasoning abilities and understandings. *Cognition and Instruction*, *28*(2), 113-145. https://www.tandfonline.com/doi/abs/10.1080/07370001003676587

Cho, Y. D. (2013). *College students' understanding of the domain and range of functions on graphs*. State University of New York at Buffalo. https://search.proquest.com/openview/d435ef37d580ebac12cc33f5b7d59213/1?pq-origsite=gscholar&cbl=18750

Cornu, B. (1981). *Advanced Mathematical Thinking*. In Nesher, P. and Kilpatrick, J. (eds), Mathematics and Cognition, 113-134.

Davis, J. D. (2007). Real-world contexts, multiple representations, student-invented terminology, and y-intercept. *Mathematical Thinking and Learning*, *9*(4), 387-418. https://www.tandfonline.com/doi/abs/10.1080/10986060701533839

Desfitri, R. (2016). In-service teachers' understanding of the concept of limits and derivatives and the way they deliver the concepts to their high school students. In *Journal of Physics: Conference Series 693*(1), 012016. https://iopscience.iop.org/article/10.1088/1742-6596/693/1/012016/meta

Diekei, R., & Isleten, T. (2004). Investigation of relation and function of some variables on the challenges J. *J. Kastamonu Education*, *11*(2), 105-116.

Drlik, D. I. (2015). Student understanding of function and success in calculus. https://scholarworks.boisestate.edu/td/893/

Dubinsky, E., & Harel, G. (1992). The nature of the process conception of function. *The concept of function: Aspects of epistemology and pedagogy*, *25*, 85-106.

Duval, R. (2006). A cognitive analysis of problems of comprehension in a learning of mathematics. *Educational studies in mathematics*, *61*(1), 103-131. https://link.springer.com/article/10.1007/S10649-006-0400-Z

Eisenberg, T. (1991). Functions and associated learning difficulties. In *Advanced mathematical thinking* (pp. 140-152). Dordrecht: Springer Netherlands. https://link.springer.com/chapter/10.1007/0-306-47203-1_9

Eisenberg, T., & Dreyfus, T. (1994). On understanding how students learn to visualize function transformations. *Research in collegiate mathematics education*, *1*, 45-68.

Forde, E. N., Robinson, L., Ellis, J. A., & Dare, E. A. (2023). Investigating the presence of mathematics and the levels of cognitively demanding mathematical tasks in integrated STEM units. *Disciplinary and Interdisciplinary Science Education Research*, *5*(1), 3. https://link.springer.com/article/10.1186/s43031-022-00070-1

Goold, E. (2012). *The role of mathematics in engineering practice and the formation of engineers* (Doctoral dissertation, National University of Ireland Maynooth). https://mural.maynoothuniversity.ie/4766

Handhika, J., Cari, C., Soeparmi, A., & Sunarno, W. (2016). Student conception and perception of Newton's law. In *AIP Conference Proceedings, 1708*, (1). https://pubs.aip.org/aip/acp/article-abstract/1708/1/070005/884476

Huang, C. H. (2010). Conceptual and procedural abilities of engineering students to perform mathematical integration. In *Proceedings of the Conference of the Joint International IGIP-SEFI Annual Conference. Trnava, Slovakei, 19*, 22. http://www.wiete.com.au/journals/WTE&TE/Pages/Vol.9,%20No.1%20(2011)/07-Huang-C-H.pdf

Kashefi, H., Ismail, Z., & Yusof, Y. M. (2010). Obstacles in the learning of two-variable functions through mathematical thinking approach. *Procedia-Social and Behavioral Sciences*, *8*, 173-180. https://www.sciencedirect.com/science/article/pii/S1877042810021294

Kultur, M. N., Ozdemir, E., & Konyalioglu, A. C. (2011). Identifying the Learning Difficulties of Freshmen in Mathematics Teacher Training Department in Function Graphs and Derivatives. *International Journal of Humanities and Social Science*, *1*(7), 119-124. https://www.academia.edu/download/113970524/16.pdf

Kurysheva, A., van Rijen, H. V., Stolte, C., & Dilaver, G. (2023). The validity, acceptability, and procedural issues of selection methods for graduate study admissions in the fields of science, technology, engineering, and mathematics: a mapping review. *International Journal of STEM Education*, *10*(1), 55. https://link.springer.com/article/10.1186/s40594-023-00445-4

Martin-Blas, T., Seidel, L., & Serrano-Fernández, A. (2010). Enhancing Force Concept Inventory diagnostics to identify dominant misconceptions in first-year engineering physics. *European Journal of Engineering Education*, *35*(6), 597-606. https://www.tandfonline.com/doi/abs/10.1080/03043797.2010.497552

Martínez-Planell, R., & Trigueros Gaisman, M. (2009). Students' ideas on functions of two variables: Domain, range, and representations. In *Proceedings of the 31st annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education, 5*, 73-80. Georgia State University Atlanta, GA. http://pegasus.uprm.edu/rmplanell/articulo2.pdf

Metcalf, R. C. (2007). *The nature of students' understanding of quadratic functions*. State University of New York at Buffalo.

https://search.proquest.com/openview/e80834c1901152addab35057c9c1cbf2/1?pq-origsite=gscholar&cbl=18750

Mohammed, A. M. (2014). *Integrated technologies instructional method to enhance bilingual undergraduate engineering students* (Doctoral dissertation, Brunel University London). https://bura.brunel.ac.uk/handle/2438/10488

Monk, S. (1992). Students' understanding of a function given by a physical model. *The concept of function: Aspects of epistemology and pedagogy*, *25*, 175-194.

Oehrtman, M., Carlson, M., & Thompson, P. W. (2008). Foundational reasoning abilities that promote coherence in students' function understanding. *Making the connection: Research and teaching in undergraduate mathematics education*, *27*, 42.

Schoenfeld, A. H. (1992). Learning to think mathematically: Problem-solving, metacognition, and sense-making in mathematics. In D. Grouws (Ed.), *Handbook for Research on Mathematics Teaching and Learning,* 334-370. New York: MacMillan.

Schwarzenberger, R. L. E. (1980). Why calculus cannot be made easy. *The Mathematical Gazette*, *64*(429), 158-166. https://www.cambridge.org/core/journals/mathematical-gazette/article/why-calculus-cannot-be-made-easy/B59C0E89F1CBBBD8A2426D57D3411785

Stewart, J. (2008). *Calculus: Early Transcendental, (6th ed)*. Thompson Brooks/Cole

Tall, D. (1992). Visualizing differentials in two and three dimensions. *Teaching Mathematics and its Applications*, *11*(1), 1-7. https://wrap.warwick.ac.uk/510/1/WRAP_Tall_dot1992b-diffls-in-2d-3d.pdf

Trigueros, M. & Martínez-Planell, R. (2009). Visualization and abstraction: Geometric representation of functions of two variables. In Proceedings *of the 29th Annual Conference of the North American Chapter of the International Group for the Psychology of Mathematics Education, T. Lamberg and L.R. Wiest, eds., University of Nevada, Reno, NV,* 100-107.

Trigueros, M., & Martínez-Planell, R. (2010). Geometrical representations in the learning of two-variable functions. *Educational studies in mathematics*, *73*, 3-19. https://link.springer.com/article/10.1007/s10649-009-9201-5

Trigueros, M., & Martínez-Planell, R. (2011). How Are Graphs of Two Variable Functions Taught? *North American Chapter of the International Group for the Psychology of Mathematics Education*. https://eric.ed.gov/?id=ED585950

Winkelman, P. (2009). Perceptions of mathematics in engineering. *European Journal of Engineering Education*, *34*(4), 305-316. https://www.tandfonline.com/doi/abs/10.1080/03043790902987378