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Tools for solving systems of non-linear equations

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Abstract

When teaching numerical techniques in courses of numerical analysis of engineering careers at the Facultad Regional San Nicolás, little emphasis is placed on the mechanical procedures of the methods. Some math programmes offer the ability to design custom graphical interfaces. Taking advantage of this possibility, the Engineering & Education Group has been developing a collection of personalised graphical interfaces related to the different topics of numerical analysis. The use of these apps in the classroom allows for generating situations where students can appreciate the importance of applying different numerical methods and discussing different involved concepts. This paper presents the apps that have been designed and a sequence of activities that will be assigned, using the apps, to learn how to obtain approximate solutions to systems of non-linear equations. Rubrics designed to analyse the degree of development of mathematical competencies that the students are expected to acquire after the proposed activities will also be shown.

Keywords: Engineering, mathematics, numerical analysis, non-linear equations, systems;

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Introduction

The solution of a system of non-linear equations is many times required when solving problems in engineering and sciences [1]–[3]. Obtaining the solutions to a system of non-linear equations such as the one that arises from the intersection of a line and a conic is an easy task, and is usually posed in analytical geometry courses in the first year of engineering careers. However, the solution of most systems of non-linear equations requires the application of numerical methods. When teaching numerical techniques in courses of numerical analysis of engineering careers at the Facultad Regional San Nicolás, Universidad Tecnológica Nacional from Argentina, little emphasis is placed on the mechanical procedures of the methods, as often too many calculations are needed to obtain good approximations. The development of other skills is pursued. Therefore, different activities requiring students to work out higher-order thinking skills, such as analysing, evaluating and creating, are incorporated into the learning material. Skills, knowledge, dispositions and values, as well as motives and motivations, are comprised of the term competence [4].

Technology can help teachers to improve class development during their teaching practices [5], [6]. The use of technological tools offers various promising possibilities, but also challenges. These tools, by themselves, cannot motivate or encourage students to learn [7]. Some math programmes offer the ability to design custom graphical interfaces. Taking advantage of this possibility, the Engineering & Education Group (GIE, Grupo Ingeniería & Educación, in Spanish) has been developing, since 2008, a collection of personalised graphical interfaces related to the different topics of numerical analysis. Initially, these tools were developed using Maple [8] or Mathematica [9] (but the applications are now programmed in SciLab [10], [11] as it is free software, reachable to every student).

SciLab can be downloaded from www.scilab.org. Introductory documentation can be obtained also in that URL. SciLab offers the possibility of creating a customised graphic user interface (GUI) that uses its computing and graphic power. With a GUI, students can interact with an algorithm, entering data, selecting options and receiving results, without the need to write the procedures that were carried out to get there. Programming the methods can be a later activity for the students. The objective of the teacher’s team numerical analysis is that students can apply the knowledge, carry out different tasks and solve problems [12]–[14].

Purpose of the study

This paper presents the apps designed with SciLab and the sequence of activities that will be assigned to learn how to obtain approximate solutions to systems of non-linear equations. The training required to use these windows is minimal because they present an intuitive interface. The use of these apps in the classroom allows for generating situations where students can appreciate the importance of applying different numerical methods and discussing different involved concepts. Also, rubrics designed to analyse the degree of development of mathematical competencies that the students are expected to acquire after the proposed activities will be shown.

Method

Systems of non-linear equations

The problem of solving a non-linear system of two equations with two unknowns can be stated as follows: Given the continuous functions *f* (*x*, *y*) and *g* (*x*, *y*), find the values $x = x^*$ and $y = y^*$ such that $f(x^*, y^*) = 0$ and $g(x^*, y^*) = 0$.

The problem is graphically shown in Figure 1 [15]. The functions *f* (*x*, *y*) and *g* (*x*, *y*) can be algebraic functions, transcendental functions or any non-linear relationship between the variables *x* and *y*. The two equations *f* (*x*, *y*) = 0 and *g* (*x*, *y*) = 0 define contour lines in the *x*–*y* plane, dividing it into regions where *f* (*x*, *y*) and *g* (*x*, *y*) is positive or negative. The solution to the problem is given by the intersection of both level curves. The number of solutions is not *a priori* known. In the graph shown in Figure 1, for example, there are four solutions. This problem is considerably more complicated than the solution of a non-linear equation.

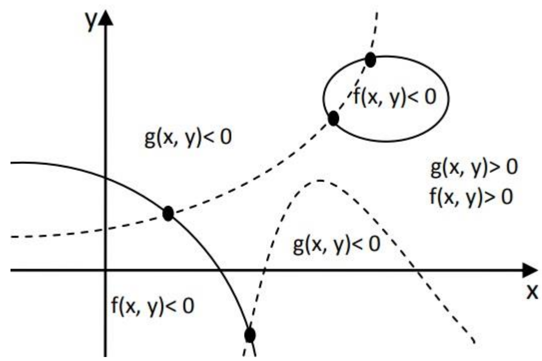


Figure 1. Graphical solution of a system of non-linear equations

Not every method used for an equation with one unknown works for non-linear systems of equations. The fixed-point iteration method and Newton's method can be extended to solve higher-order systems. These methods will be presented below, for non-linear systems of two equations with two unknowns.

The theoretical details, which are supposed to be known, will not be discussed in this work [16]–[19].

Fixed-point iteration method

In a non-linear system of *n* equations with *n* unknowns, *F*(*x*) = 0:

$$\begin{cases} f_1(x_1, x_2, x_3, \dots, x_n) = 0 \\ f_2(x_1, x_2, x_3, \dots, x_n) = 0 \\ \cdots \\ f_n(x_1, x_2, x_3, \dots, x_n) = 0 \end{cases} \tag{1}$$

can be rearranged by solving the *i*-th equation for x_i, so that the system becomes a fixed-point problem:

$$\begin{cases} x_1 = g_1(x_1, x_2, x_3, \dots, x_n) \\ x_2 = g_2(x_1, x_2, x_3, \dots, x_n) \\ \dots \\ x_n = g_n(x_1, x_2, x_3, \dots, x_n) \end{cases} \tag{2}$$

To approximate the fixed point of *G*, an initial approximation must be chosen. The algorithm consists in calculating the functions g_1, g_2, …, g_n in the *i*-th approximation, to obtain the approximation (*i* + 1)th. Under conditions of existence and uniqueness of the solution, this sequence converges to the fixed point of *G*, which is the solution of $F(x) = 0$.

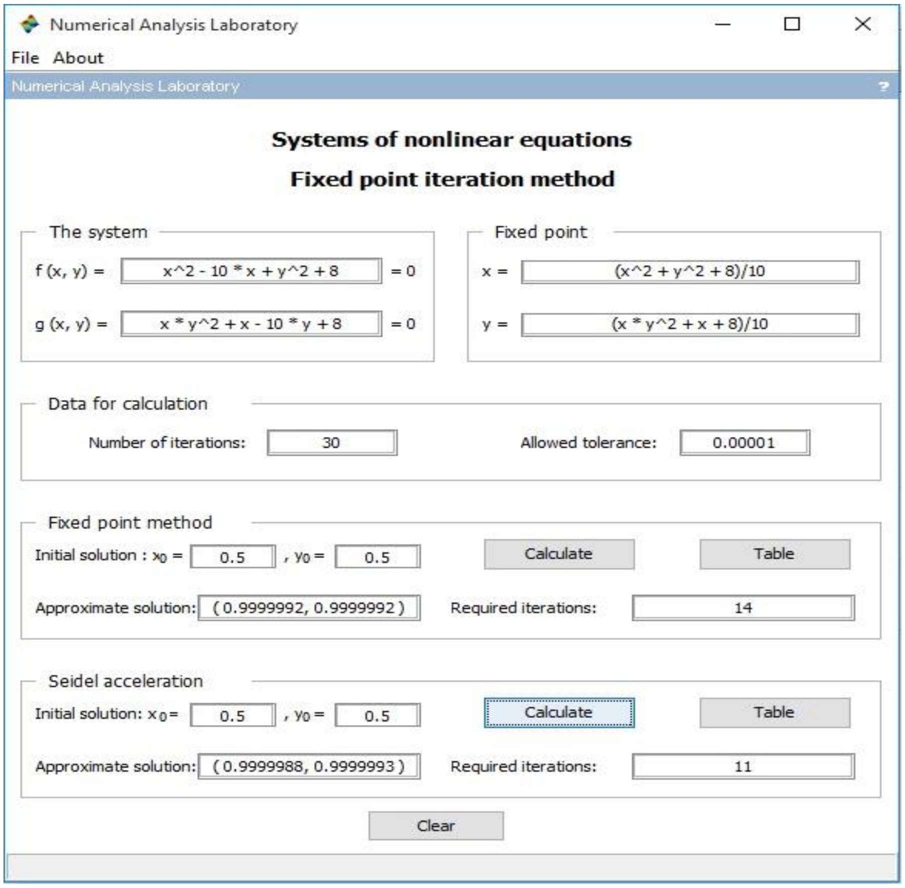
One way to speed up the convergence of the fixed-point iteration method is to use the most recent estimates of the components x_1, x_2, …, x_{i-1} to compute x_i. This way of obtaining the approximations is called Seidel acceleration.

Figure 2 shows the app designed with SciLab to obtain the approximation of the solution of a non-linear system of two equations with two unknowns. Both *F* and *G* must be written, along with the maximum number of iterations to be done and the allowed tolerance, as data for doing the calculations.

Consider the system:

$$\begin{cases} x^2 - 10x + y^2 + 8 = 0 \\ xy^2 + x - 10y + 8 = 0 \end{cases} \tag{3}$$

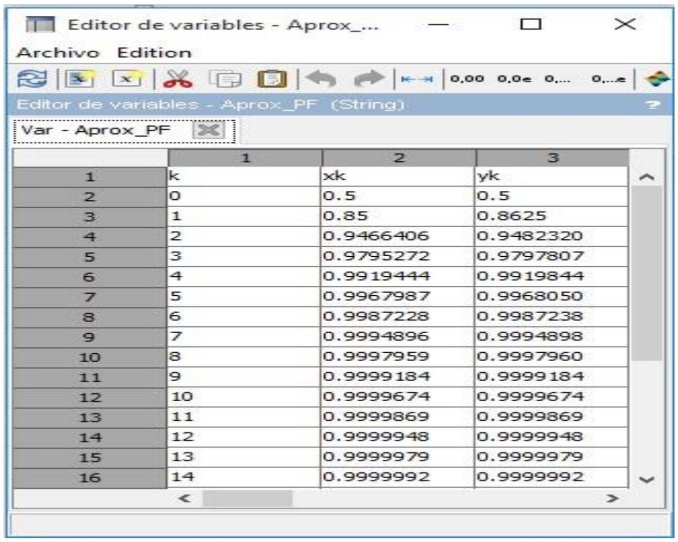
Starting from the same initial approximation $x^{(0)} = (0.5, 0.5)$ and applying the fixed-point iteration method and the Seidel acceleration, the results shown in Figure 2 are achieved. The application also displays the number of iterations required to obtain the solution with the allowed tolerance. If the desired approximation cannot be achieved with the number of iterations or the allowed tolerance, a message appears informing the user that the number of iterations is not enough.



Figures 2. Tool designed for using the fixed-point iteration method

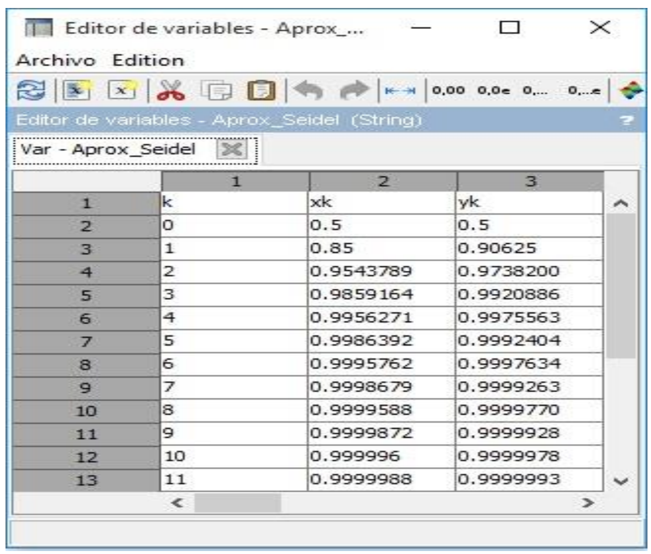
This tool also provides the possibility of displaying the complete sequence of calculations using the Table button, after having pressed the Calculate button.

Figures 3 and 4 show the sequences obtained by applying the fixed-point iteration method and the Seidel acceleration, respectively, for the example shown in Figure 2.



	1	2	3
1	k	xk	yk
2	0	0.5	0.5
3	1	0.85	0.8625
4	2	0.9466406	0.9482320
5	3	0.9795272	0.9797807
6	4	0.9919444	0.9919844
7	5	0.9967987	0.9968050
8	6	0.9987228	0.9987238
9	7	0.9994896	0.9994898
10	8	0.9997959	0.9997960
11	9	0.9999184	0.9999184
12	10	0.9999674	0.9999674
13	11	0.9999869	0.9999869
14	12	0.9999948	0.9999948
15	13	0.9999979	0.9999979
16	14	0.9999992	0.9999992

Figure 3. The sequence obtained with the fixed-point method



	1	2	3
1	k	xk	yk
2	0	0.5	0.5
3	1	0.85	0.90625
4	2	0.9543789	0.9738200
5	3	0.9859164	0.9920886
6	4	0.9956271	0.9975563
7	5	0.9986392	0.9992404
8	6	0.9995762	0.9997634
9	7	0.9998679	0.9999263
10	8	0.9999588	0.9999770
11	9	0.9999872	0.9999928
12	10	0.999996	0.9999978
13	11	0.9999988	0.9999993

Figure 4. The sequence obtained with the Seidel acceleration

Newton's method

Consider a non-linear system of two equations with two unknowns such as:

$$\begin{cases} f(x,y) = 0 \\ g(x,y) = 0 \end{cases} \quad (4)$$

Suppose an approximate solution of this system, (x_i, y_i), is known. Then, writing a Taylor series expansion of the functions f and g around this point and evaluating these expressions at (x^*, y^*), assuming that this point is a solution to the system:

$$f(x^*,y^*) = f(x_i,y_i) + f_x|_{(x_i,y_i)}(x^* - x_i) + f_y\big|_{(x_i,y_i)}(y^* - y_i) + \cdots = 0 \quad (5)$$

$$g(x^*,y^*) = g(x_i,y_i) + g_x|_{(x_i,y_i)}(x^* - x_i) + g_y\big|_{(x_i,y_i)}(y^* - y_i) + \cdots = 0 \quad (6)$$

Neglecting in expressions (5) and (6) the terms that are after the first derivatives (the error) and rearranging terms, a system of linear equations with *x** and *y** as unknowns is obtained (7):

$$\begin{aligned} f_x|_i(x^* - x_i) + f_y\big|_i(y^* - y_i) &= -f_i \\ g_x|_i(x^* - x_i) + g_y\big|_i(y^* - y_i) &= -g_i \end{aligned} \tag{7}$$

where $f_i = f(x_i, y_i)$ and $f_x|_{(x_i,y_i)} = f_x|_i, f_y\big|_{(x_i,y_i)} = f_y\big|_i$ and similarly for g. Solving this system, a new approximation of the solution is obtained.

Consider the system:

$$\begin{cases} x^2 + y^2 - 2x - y = 0 \\ \frac{x^2}{4} + y^2 - 1 = 0 \end{cases} \tag{8}$$

Starting from the initial solution $x^{(0)}$ = (1, 1) and applying Newton's method, the results shown in Figure 5 are achieved. Similar to the previously presented tool, this app displays the number of iterations required to obtain the solution with the allowed tolerance and allows one to see the complete sequence of calculations using the Table button, after pressing the Calculate button.

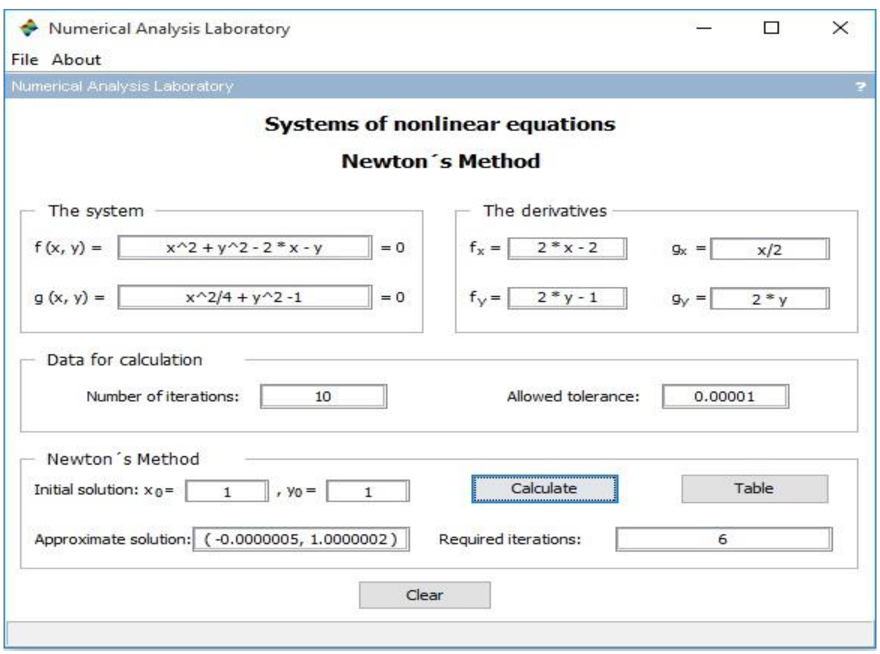
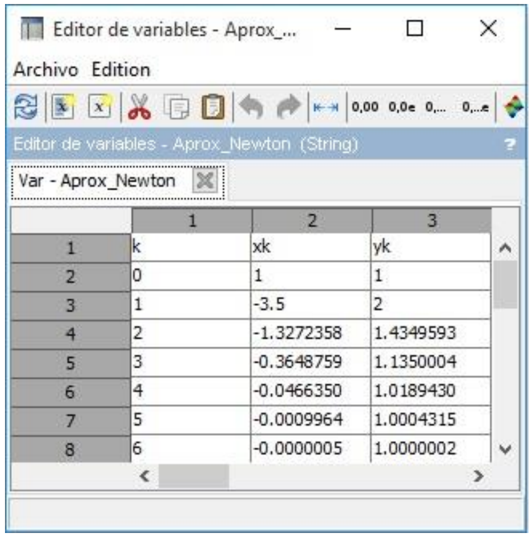


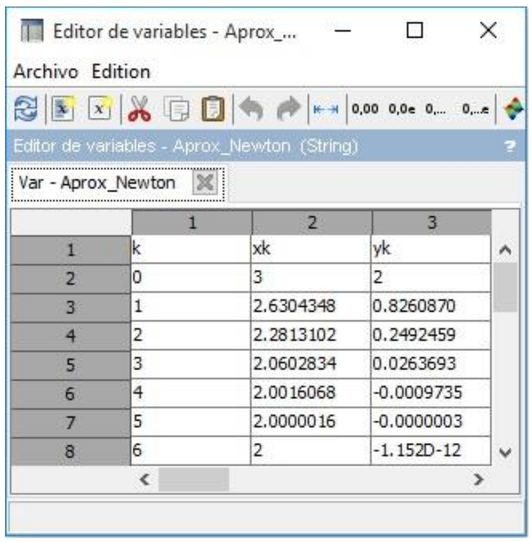
Figure 5. Tool for using Newton's method

Figures 6 and 7 show the sequences obtained by applying Newton's method for the system (8) and the initial solutions $x^{(0)}$ = (1, 1) and $x^{(0)}$ = (3, 2), respectively. It is observed that two different solutions are obtained: (0, 1) and (2, 0).



	1	2	3
1	k	xk	yk
2	0	1	1
3	1	-3.5	2
4	2	-1.3272358	1.4349593
5	3	-0.3648759	1.1350004
6	4	-0.0466350	1.0189430
7	5	-0.0009964	1.0004315
8	6	-0.0000005	1.0000002

Figure 6. The sequence obtained with $x^{(0)} = (1, 1)$



	1	2	3
1	k	xk	yk
2	0	3	2
3	1	2.6304348	0.8260870
4	2	2.2813102	0.2492459
5	3	2.0602834	0.0263693
6	4	2.0016068	-0.0009735
7	5	2.0000016	-0.0000003
8	6	2	-1.152D-12

Figure 7. The sequence obtained with $x^{(0)} = (3, 2)$

As a linear system has a solution whenever the determinant of the coefficient matrix (the Jacobian matrix of *F* evaluated in (x_i, y_i) is not null, problems arise when this occurs in any iteration. When the initial solution is not properly chosen and the Jacobian in an iteration becomes null, a message like the one shown in Figure 8 appears.

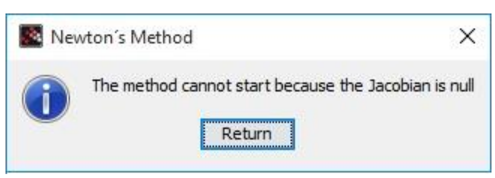


Figure 8. Message for wrong initial solution

Results

Using the tools to assess students

After presenting the tools and having solved some examples with students on paper, the following is an example of an activity to carry on with students in class, to enforce the understanding of the methods for solving non-linear equations.

Two of the solutions of the non-linear system given by (9) are placed in [−1,1] × [−1,1] [19]. Obtain an approximation of each one, using both fixed-point iteration and Newton's method, and discuss the obtained solutions.

$$\begin{cases} 5x^2 - y^2 = 0 \\ y - 0.25(sin(x) + cos(y)) = 0 \end{cases} \tag{9}$$

First, students will be asked to make a graphic representation of the system, in order to make a first estimation of the solution. They can use any tool, like Geogebra, Symbolab or WolframAlpha. For example, the graph shown in Figure 9 was obtained in WolframAlpha, with the blue function corresponding to the first equation and the red to the second one.

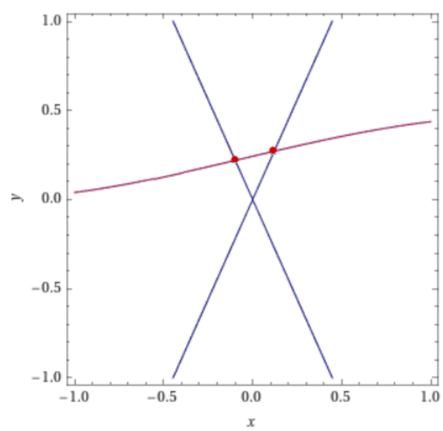


Figure 9. Graphical representation to estimate the solutions of the system

Based on the graph, it could be concluded that the first approximation for one of the solutions may be considered as (−0.1, 0.2), and (0.1, 0.2) as the first approximation for the other solution.

Then, students will be asked to apply fixed-point iteration to obtain better approximations of each solution. To do this, they will have to obtain proper expressions, such as $x = g_1(x, y)$ and $y = g_2(x, y)$, for each solution, and show that the existence and uniqueness conditions are satisfied in a proper domain containing each solution in each case. It is an easy task. It must be noticed that the expression for *x* is not the same for the two points, since when clearing the variable *x* of the first equation there is a square root and the sign must be taken into account. Figures 10 and 11 show the results obtained, and the number of iterations needed in each case, together with the first terms of the sequences. In these figures, the expression of the square root with the corresponding sign in each case can be observed.

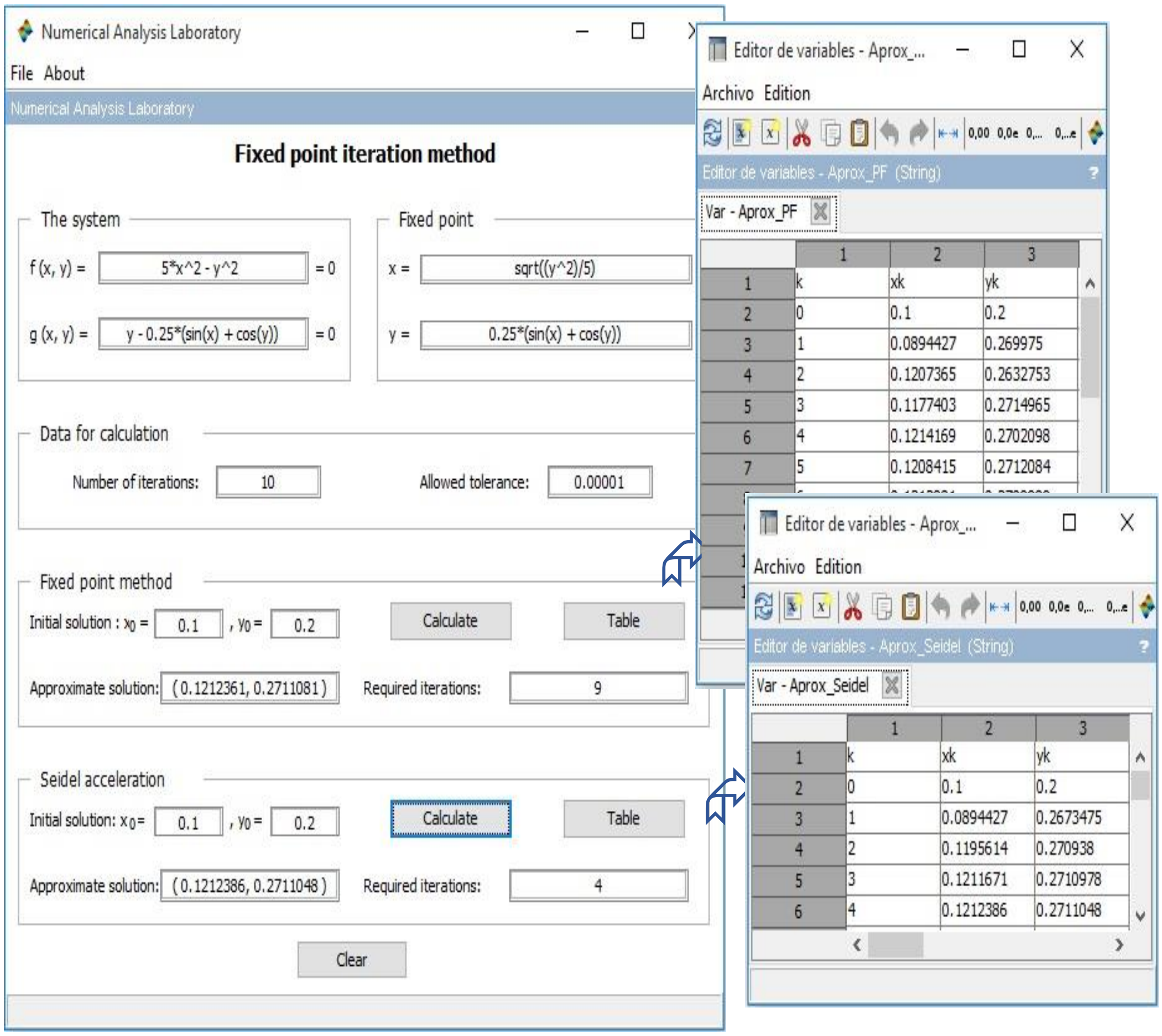


Figure 10. Using the fixed-point tool to estimate the solution of the system in the first quadrant

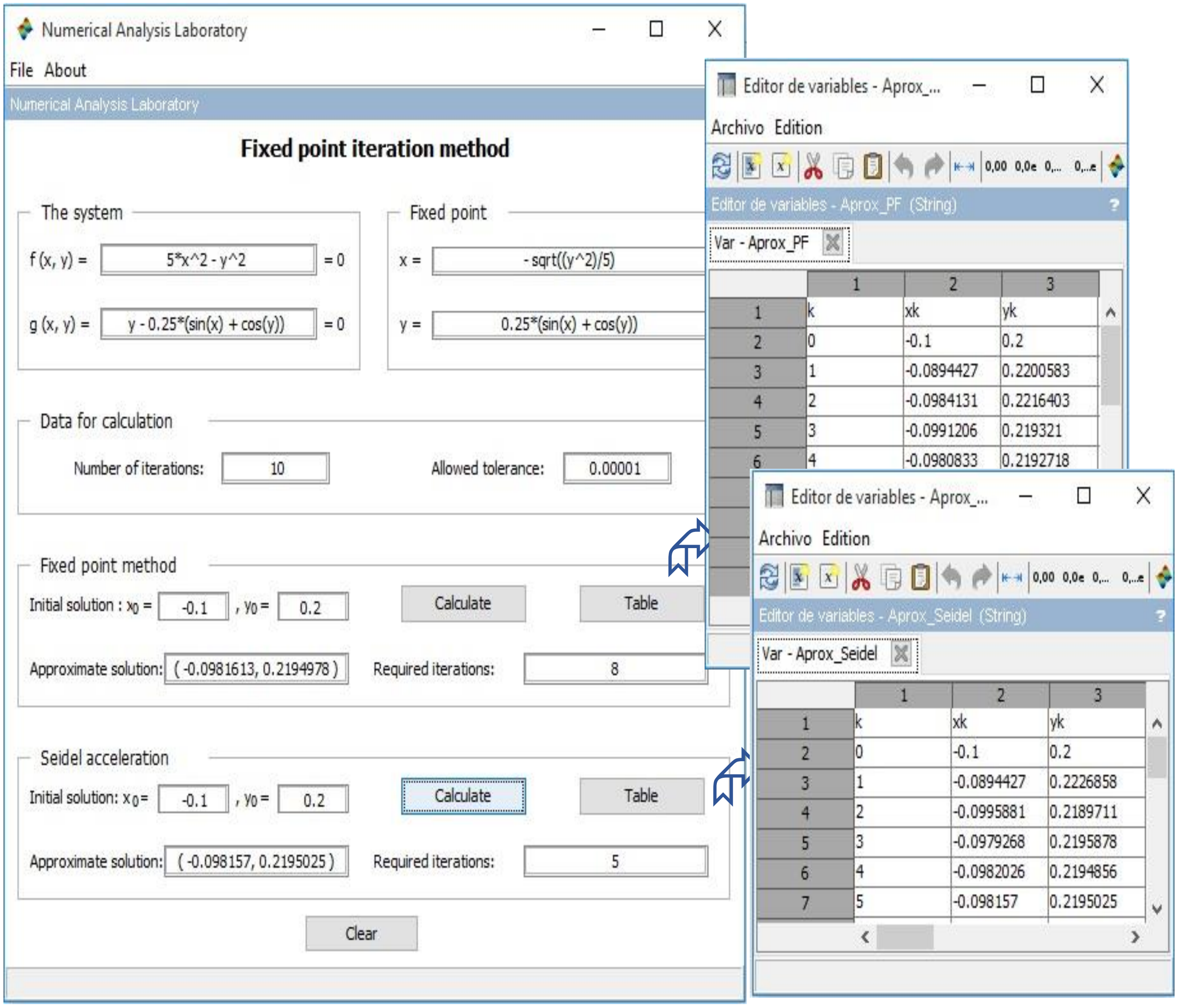


Figure 11. Using the fixed-point tool to estimate the solution of the system in the second quadrant

It should be noted that as the stopping criterion deals with the absolute value of the functions involved near cero, they will not find the number of exact decimal digits as the tolerance indicates.

Furthermore, continuing with the activity, the next task will be to apply Newton's method to approximate both solutions, using the second tool. Now the functions involved in the original equations should be used, starting with each of the approximations obtained graphically. Also, the derivatives of the functions must be entered.

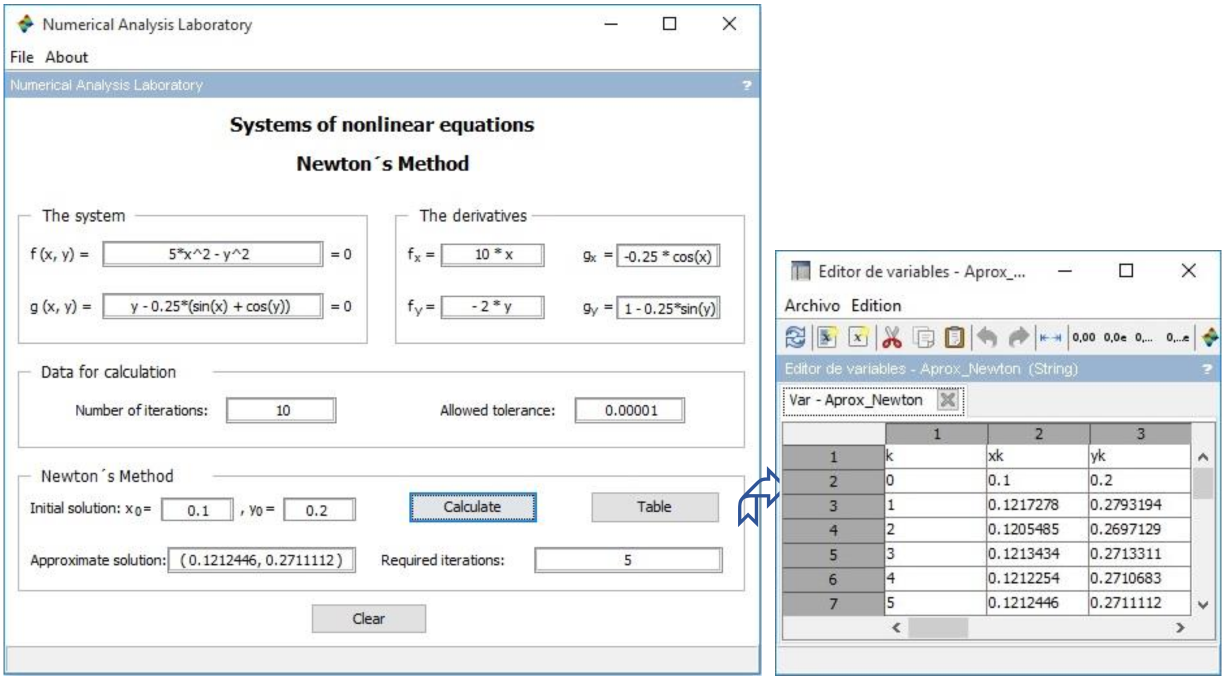


Figure 12. Using Newton's method to estimate the solution of the system located in the first quadrant

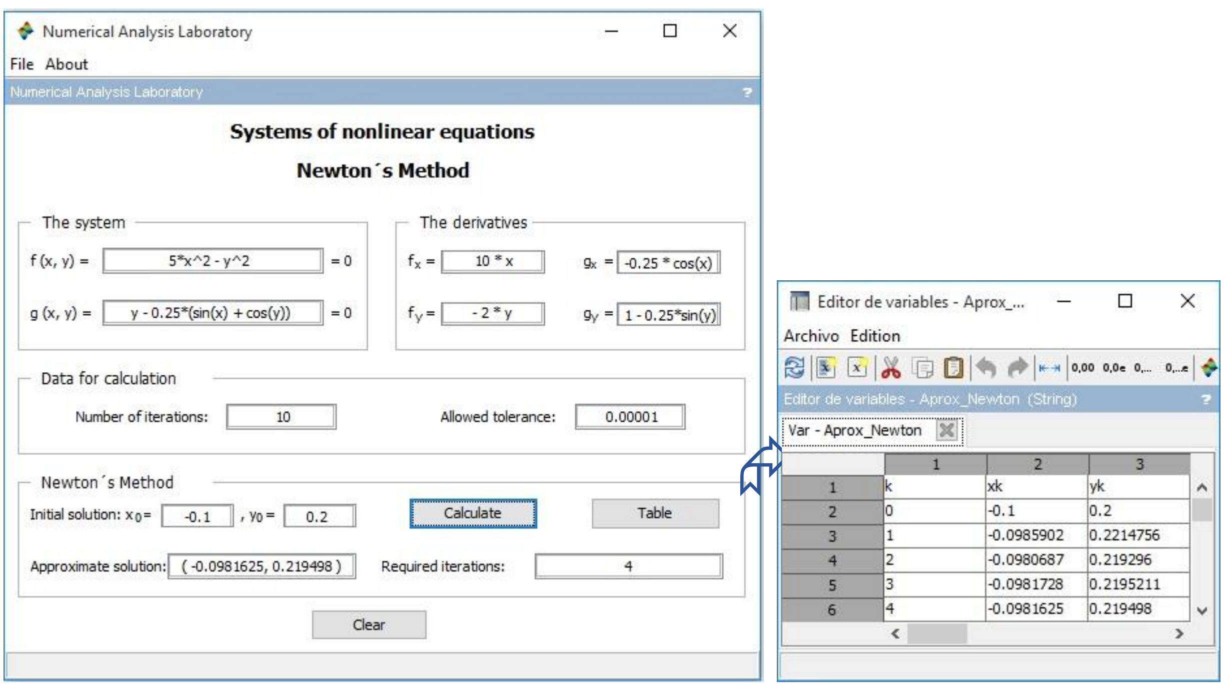


Figure 13. Using Newton's method to estimate the solution of the system located in the second quadrant

After using the tools, students can summarise the results obtained, as shown in Tables 1 and 2, and compare the number of iterations needed in each case.

Table 1. Approximations for the solution located in the first quadrant

Method	Approximation	Iterations needed
Fixed point iteration	(0.1212361, 0.2711081)	9
Fixed point + Seidel acceleration	(0.1212386, 0.2711048)	4
Newton	(0.1212446, 0.2194978)	7

Table 2. Approximations for the solution located in the first quadrant

Method	Approximation	Iterations needed
Fixed point iteration	(−0.0981613, 0.2194978)	8
Fixed point + Seidel acceleration	(−0.098157, 0.2195025)	5
Newton	(−0.0981625, 0.219498)	4

Analysing the results, students can see that both methods give a good approximation for the two solutions, in less than 10 approximations. And watching the list of the approximations given by the Table button of the tools, it can be seen that in three iterations, a precision of three decimal digits is obtained in four iterations.

The rubric for the assessment

A rubric can be defined as an assessment tool used to measure the achievement of learning outcomes in the classroom considering a consistent set of criteria. The analytical rubrics are double-entry tables where the evaluation criteria are located in the rows and the domain levels in the columns. Nowadays, rubrics are a common way of sharing explicit assessment criteria with students. Rubrics homogenise the evaluation criteria for both teachers and students [20]–[22] and may help students to understand teachers' expectations and feedback [21]–[24].

Table 3 shows the analytical rubric designed to assess an activity to be carried out by students when solving an exercise similar to the one presented in the previous section.

Table 3. Analytical rubric

	Needs to improve	Regular	Good
Obtaining the graphical solution (20%)	The obtained graph does not show the required solutions	The obtained graph poorly shows the required solutions	The obtained graph clearly shows the required solutions
Using the tool for fixed point iteration (30%)	No solution was obtained by FP iteration	Only one of the solutions was correctly obtained by FP iteration	The two solutions were correctly obtained by FP iteration
Using the tool for Newton's method (30%)	No solution was obtained by Newton's iteration	Only one of the solutions was correctly obtained by Newton's iteration	The two solutions were correctly obtained by Newton's iteration
Discussion (20%)	No analysis was presented	A comparison of the solutions obtained, and the number of iterations needed was presented	Comparison of the solutions and number of iterations required, advantages and disadvantages of the methods were stated, together with the use of the apps

This rubric should be available for students when the activity is assigned to let them know which issues they have to consider when they work on it.

Using rubrics like the one presented in this paper helps to carry on an impartial, objective and uniform correction by a team of teachers.

Conclusion

The kind of tools presented in this paper allow students to concentrate on the analysis of the results given by the numerical methods used, rather than spending time doing endless calculations or writing codes when they do not master programming languages. To avoid falling into routine and mechanical activities with the use of these tools, problems and exercises that require analysis of the results should be selected appropriately, generating dynamic learning sequences, where the results are not as expected, and generating discussion environments in the class.

When using rubrics, students must get in contact with the assessment tool before they carry out the activity. It helps them to focus on the items that will be evaluated and, consequently, on what was meant to be learned.

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