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Mathematical discourses on propositional equivalence: An exploration through the commognitive lens

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Abstract

This paper describes the mathematical learning of purposively sampled second-year Bachelor of Secondary Education Mathematics majors from a state university in Bulacan province on the propositional equivalence concept within the Logic and Set Theory course via the commognitive lens. This small-scale study employed exploratory qualitative research with one class recording, one focus group and select activity outputs. Four participants in the focus group were sampled based on commognitive conflict occurrences. The teacher-researcher operated as a co-participant in the mathematical discourse. The dean's approval and participants' informed consent were observed, explaining the research objectives and confidentiality scope. The findings present accounts and descriptions of participants' mathematical discourses through the commognitive lens: word use, visual mediators, endorsed narratives and routine practices that describe the Logic and Set Theory discourses on the propositional equivalence concept from a participationist's learning standpoint.

Keywords: Commognition, logic, mathematical discourse, propositional equivalence;

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1. Introduction

One of the Sustainable Development Goals agreed upon by world leaders is quality education (UNESCO, 2021). It entails equitable access to quality education, such as proper skills and competencies development issues. Competencies are clear descriptions of what the graduates can demonstrate after completion of a specific process of learning (Schiersmann et al., 2016; Zuckerman, Azari, & Doane, 2013), which can be categorised into three levels: institutional, programme and course (Biggs & Tang, 2007). Following this, one of the outcome statements at the programme level for Bachelor of Secondary Education Mathematics majors is to exhibit competencies or the ability to explain and demonstrate mathematical concepts and procedures clearly (Commission on Higher Education, 2017). From the curriculum components of the Bachelor of Secondary Education Mathematics programme, the newly added Logic and Set Theory course equips students with the said competency.

Logic centres on the procedures and principles of reasoning refine mathematics as a learning area (Koshy, 2004; Research Centre for Teacher Quality, 2020). In logic, theorems are the backbones of mathematics (Koshy, 2004). A theorem is a proposition that can be shown to be true using proof (Rosen, 2012). A proposition, in mathematics, is a declarative sentence that is either true or false (Sundstrom, 2014). Proofs are a set of arguments that are substantial evidence of the validity of the mathematical theory. An argument, in mathematical logic, is a structure of propositions directed at establishing the truth of an assertion. The proposition found at the end is the conclusion, while the preceding propositions are called premises (Epp, 2011; Sundstrom, 2014). A vital step in a mathematical argument is replacing a proposition with another with the same truth values, demonstrating *propositional equivalence* (Rosen, 2012). Establishing propositional equivalence is structural in learning mathematical arguments and proof methods (Velleman, 2006). In any mathematical theory, new words are defined using those previously defined. However, this process has to start somewhere (Epp, 2011). Therefore, any mathematical objects used and defined in basic or advanced mathematics are developed, abstracted and realised through human imagination (Bullock, 1994). Undeniably, college mathematics deals with various mathematical objects (Velleman, 2006). According to Koshy (2004), mathematics is a concise language with vocabulary, symbolism and rules that train students to express themselves precisely and concisely. Mathematics could not only be thought of as an expansion of the current language but rather as a separate and new language because it has its own set of structural rules that define how mathematical words, objects, narratives and routines are organised and constructed (Bullock, 1994; Ioannou, 2018). If one understands the language, one will know what the newly defined word is and precisely what can be said about that word.

For this reason, to understand students' mathematical discourse and cognition on establishing propositional equivalences within the Logic and Set Theory course, it will be helpful to study the discourse patterns and ambiguities through their visual mediators, words used, endorsed narratives and routines. Thus, the main objective of this paper is to explore and describe the mathematical learning of the Bachelor of Secondary Education Mathematics majors on the propositional equivalence concept. This study contributes to the commognition research in mathematics in the Philippines.

2. Literature review

2.1. Commognitive framework

A discursive approach to mathematics education is typically branded as 'participationist' (Sfard, 2007). The views change in an individual as a result of the communalisation of the individual and the individualisation of the collective (e.g., solving mathematical problems means a continuing transition from participating in a group task to becoming proficient in carrying out the task in its entirety on their own) (Nardi, Ryve, Stadler, & Viirman, 2014; Sfard, 2007). Interpersonal communication and individual cognition are two portions of the same thing (Ho, Lim, Tay, Leong, &

Teo, 2019). This perspective promotes and executes mathematics teaching and learning discourses (Mpofu & Mudaly, 2020). Accordingly, the term '*commognition*' (i.e., from the words 'communication' and 'cognition') was created to cover thinking and communicating to stress the harmony of the two processes (Lestari, Nusantara, Susiswo, Chandra, & Indrawatiningsih, 2021; Sfard, 2008). In the commognitive framework, discourse is considered the principal object of attention (Martin-Molina, González-Regeña, Toscano, & Gavilán-Izquierdo, 2020) and a determiner that includes or excludes individuals from a stated discourse (Ärlebäck & Frejd, 2013). Thinking is conceptualised as a discursive activity or an evolving form of communication (Sfard & Lavie, 2005), an individualised form of 'interpersonal' communication and stating self-communication (Emre-Akdoğan, Güçler, & Argün, 2018). Mathematics is conceptualised as a discourse form and learning is equivalent to extending and modifying the discourse, indicative of discourse engagement (Viirman, 2015). Mathematical thinking gradually develops from interpersonal communication about mathematical objects, implying collective efforts (Sfard, 2007, 2008, 2018b).

To decide whether a particular discourse is 'mathematical', the indicators are as follows: visual mediators, word use, routines and endorsed narratives (Nardi et al., 2014; Sfard, 2018a). *Word use* is tantamount to word meaning (Viirman, 2014). This includes mathematical terminology and familiar words with a definite connotation within mathematics with four developing phases. *Passive use* means that the mathematical word is not uttered from the participants' speech; *routine-driven use* denotes that mathematical words only about specific but limited procedures and actions they perform are uttered; *phrase-driven use* denotes that complete phrases rather than the word create the structure of utterances; and lastly, *object-driven use* denotes that the mathematical words, i.e., as if they refer to objects, new results or end states, are uttered (Emre-Akdoğan et al., 2018). *Visual mediators* are channels with which participants are familiar with the object of conversation and establish their discourses (e.g., logic symbols $\rightarrow, \vee, \leftrightarrow, \wedge, \equiv$). Literate mathematical discourses make massive use of symbolic objects for mathematical communication purposes. *Endorsed narratives* are written or spoken text, drawn as objects' descriptions relating to objects or activities with objects, subject to rejection or endorsement, that is being regarded as true or false (e.g., theorems, definitions and proofs). *Routines* are precise practices in participants' actions used in distinctive ways by the community (e.g., defining and proving) (Park, 2017; Roberts & Le Roux, 2019; Sfard, 2007).

Routines can be divided into three: exploration, deeds and rituals. *Exploration* aims to advance discourse through producing, developing or verifying endorsable narratives. Explorations are divided into three types: construction, substantiation and recall. *Construction* aims at creating new endorsed narratives; *substantiation* aims at deciding whether to endorse previously created narratives; *recall* aims to call upon narratives endorsed in the past. *Deeds* aim to change the actual objects, which produce a physical change, including physical entities and written work. *Rituals* aim to create and sustain social approval with other participants in the mathematical discourse. Rituals' goal is neither to produce endorsed narratives nor physical changes but rather to establish a social bond between the interlocutors (Daher, 2020; Lavie, Steiner, & Sfard, 2019; Sfard, 2008). Lavie et al. (2019) describe a routine as *practical*, if a person interprets a task situation as requiring a change, re-organisation or re-positioning of objects, and as *discursive*, if a person interprets a task situation as requiring a communicational action.

2.2. Communalisation of the individual

Discourse patterns are the consequences of processes managed by rules. These rules can either relate to object properties of the discourse or activity structures that produce and support narratives. *Object-level rules* are rules that are connected directly to the objects' definitions. On the contrary, *metalevel rules* relate to the proof or substantiation process of new (i.e., to novice students) mathematical results (Ioannou, 2018). Learners' discourse can either be ritualised or explorative. *Ritualised discourse* includes the learner following or imitating strict rules regulated by a more fluent participant. *Explorative discourse* is a more complex form characterised by narratives on mathematical objects endorsing mathematical axioms, definitions and theorems (Roberts & Le Roux, 2019). Other occurring discourses in the commognitive are *colloquial discourses* filled with familiar everyday words from personal experiences; *classroom discourses* filled with school norms; and *literate discourses* filled with specialised symbolic artefacts (Ärlebäck & Frejd, 2013).

2.3. Individualisation of the collective

The relationship between object-level and the metalevel cognitive processes was expected to include two higher order metacognitive processes: control and monitoring (Nelson & Narens, 1994). *Monitoring* signifies the process by which the metalevel is informed about what is occurring at the object level, whereas *control* refers to the processes and mechanisms by which the metalevel regulates and makes the operation of object-level processes conform to the achievement of a specific goal (Koriat, 2019; Nonose, Kanno, & Furuta, 2012).

2.4. Commognitive conflict

Mathematical communication signifies 'effective' if each interlocutor has no basis or reason for suspecting a breach. The circularity of the mathematical discourse development process and the ambiguity of the nature of its objects pave the way for the notion of *commognitive conflict*. In the mathematical discourse of the participants, this conflict is pondered as a usual encounter on the same mathematical tasks symptomatic of and conforming to disagreeing rules or using the same symbols or objects in conflicting ways (Sfard, 2008). Commognitive conflict is an entrance to a new discourse rather than a hindrance; genuine commitment to overcome hurdles must be present to old-timers and newcomers (Ioannou, 2018; Pratiwi, Nusantara, Susiswo, & Muksar, 2020).

3. Methodology

3.1. Design and participants

This qualitative study employed exploratory research (Stebbins, 2008). It refers to methodical data collection intended to derive generalisations inductively about a phenomenon under study. The sampling design used is purposive (Fraenkel & Wallen, 2009). The participants were undergraduate teacher education students majoring in mathematics at a state university in Bulacan, Philippines, taking the mathematics course Logic and Set Theory the first semester of the school year 2021–2022. The teacher-researcher served as a co-participant in the discourse. Of 39 students, 4 students were selected to include in the focus group discussion (FGD). Based on the commognitive conflicts evident in the class discussions and select activity output, these participants were selected.

3.2. Tools and procedures

The data used in this study are from a video recording of class discussion, FGD and select student outputs. The first data is the video recording. One class video recording (i.e., 2 hours and 5 minutes) comprising a topic on propositional equivalence was utilised. The following data is the FGD data. FGD was conducted and documented with four select participants to acquire relevant ideas and abstractions. These participants were selected based on the commognitive conflict occurrences evident in the class discussions and select activity outputs. The teacher-researcher focused on the learning aspects of the math discourses of the participants. Class and FGD recordings were transcribed and translated, approved by a language expert and later sorted, as triangulated by student activity outputs, based on the commognitive lens. The type of discussion used is a discussion where questions are predetermined but can still be developed following the study's objectives so that the teacher-researcher still has control over the topic for the discussion. The teacher-researcher explored by asking questions to explore the answers to the problems that have been determined, particularly in the context of the commognitive. The teacher-researcher's data analysis examined the mathematical discourses in the framework of the commognitive revealed in the students' responses in the discussion, FGD and the scripts of relevant activities. The teacher-researcher asked for approval from the college dean to conduct the study and moved to the actual data collection after getting permission. Informed consent, respect for confidentiality and anonymity and voluntary participation were observed. The researcher clarified all vital matters about the study to the participants and confirmed voluntary involvement using a consent form. Thus, participants must have voluntarily filled out the consent form before participating in the study – defining the entire informed consent process.

4. Results and discussions

A vital step in a mathematical argument is replacing a proposition with another with identical truth values. The class discussion started (Table 1) with the definition of *logical equivalence* (Figure 1). The module comprising notes related to the lesson 'propositional equivalences' was given to the students ahead of time as reading materials.

Table 1. Class discussion excerpt A

[4] Teacher:	Okay. Definition 1.2.1. Let *P* and *Q* be propositional forms. What did you notice with the letters I have used before (in our previous sessions) and the letters *P* and *Q*? What have you observed?
[5] Student:	It's… a big letter.
[6] Teacher:	Last meeting, I used a small letter. What does it mean? What does a small letter denote? Now, here in my example, I have here a big letter or a capital letter. What do these capital letters denote? They represent what?
[7] Student:	The big or the capital letters represent propositional forms.
[8] Teacher:	How about small (letters)?
[9] Student:	They're propositional variables.

At this point, the teacher-researcher asked the students about the *visual mediators* used. It can be observed that the words used in the discourse are *routine-driven*. When asked about the differences between the mathematical objects, the words moved into *phrase-driven use* (Table 2).

Table 2. Class discussion excerpt B

[12] Teacher:	Again, what is a propositional variable?
[13] Student:	It represents a propositional statement.

While the propositional variable denotes a proposition or statement, using the word '*propositional statement*' is unclear. In this case, the student's *endorsed narrative* concerning the definition of the propositional variable can be *rejected*. The teacher-researcher then used this commognitive conflict to access a new discourse (Table 3), highlighting the terms *proposition*, *compound proposition*, *statement*, *propositional variable* and *propositional form*.

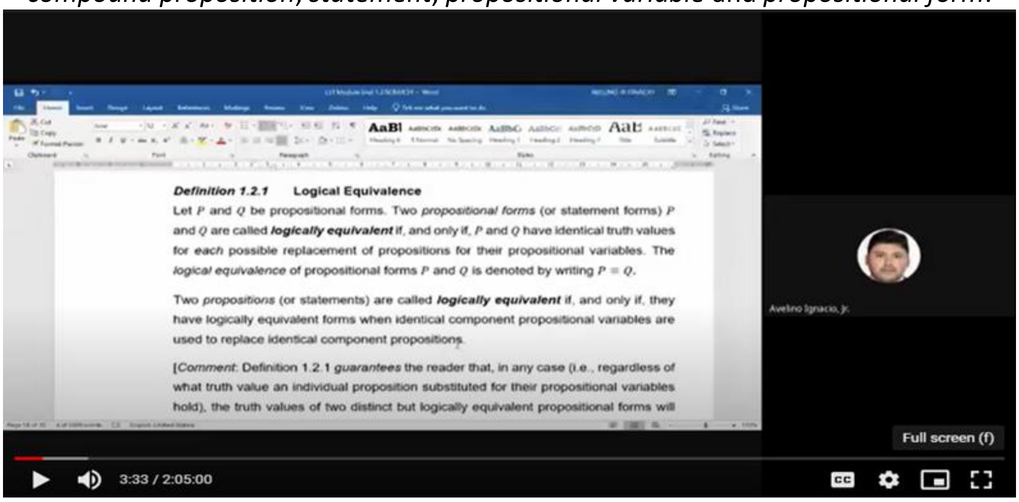

Figure 1. Definition of logical equivalence

Table 3. Class discussion excerpt C

[16] Teacher:	What is a propositional form?
[19] Student:	On the propositional form, they are purely symbolic of the compound proposition, and when we say propositional form, we use propositional variables and logical operators.
[23] Teacher:	Let capital letters *P* and *Q* be propositional forms and are logically equivalent. When can you say that *P* and *Q* are logically equivalent?

To usher the class to the concept of logical equivalence, Example 1 (Figure 2) was posted. The students were initially asked to identify how many propositional forms are there and then state the truth value/s represented by their propositional variables (Table 4).

Table 4. Class discussion excerpt D

[34] Teacher:	Do you know what is the truth value being represented by *P*? And what is the truth value being represented by *Q* here? … By simply looking at example 1, where *P* is the propositional form, … uhm, do you know the truth value of *p*? …
[35] Student:	My answer, Sir, is NO.
[37] Student:	(Sir), How can we determine the truth value if there is no statement?

Example 1

P: $\neg(p \wedge q)$

Q: $(\neg p \vee \neg q)$

$$\neg(p \wedge q) \equiv (\neg p \vee \neg q)$$

p	q	$\neg p$	$\neg q$	$p \wedge q$	$\neg(p \wedge q)$	$(\neg p \vee \neg q)$
T	T	F	F	T	F	F
T	F	F	T	F	T	T
F	T	T	F	F	T	T
F	F	T	T	F	T	T

Figure 2. The first example of propositional forms

It can be observed that the occurring discourse in the commognitive is a *classroom discourse* filled with school norms and not just a colloquial discourse. Since they do not know what the propositional variables p and q represent (i.e., whether true or false) in the propositional form P, they thought of each possible replacement and concluded the number of combinations P can produce – four. From each combination, the truth values of P and Q were found and compared. It was found identical and inferred that P and Q are logically equivalent (Table 5).

Table 5. Class discussion excerpt E

[52] Teacher:	Now, (for) each possible replacement: true–true, true–false, false–true, false–false. For all possible propositions that we could replace, in light of the truth values, the endmost result must be the same.

[74] Teacher:	So, these (statements). What can we say about this?
[75] Student:	*P* and *Q* are logically equivalent.

In attempting to learn the concept of 'logical equivalence', the communalisation of the individual presented a *ritualised classroom discourse*, indicative of developing *object-driven word use*. It shows a *ritual discursive routine*. To show propositional equivalence, i.e., the logical equivalence of propositional forms, it is clear to all the participants the importance of the use of a truth table in establishing such. The teacher-researcher offered a summary of laws of logical equivalences (Figure 3). Participants initially developed an impression that the opposite of the conjunction is the disjunction due to the logical equivalence presented by De Morgan's law. Although this may sound 'correct' for propositional forms with a structure similar to De Morgan's law, this must be done with caution. In advance, the participants must perceive the structure of De Morgan's law and apply the law properly. This scenario provided the participants access to enter a new discourse. To establish the relevance of substantiation and construction routines, the participants were asked whether there was still a need to reason out each step using a series of logical equivalences. The level of conversation is more of a *colloquial* to *classroom discourse* with the *phrase-driven word use*. Although it may not sound like literate discourse due to the absence of massive mathematical objects, it cannot be denied that a wide-ranging view must be seen to establish the relevance of the substantiation and construction routines (Table 6).

Summary of Logical Equivalences (A Special Case)

Name	*Equivalence*
Identity Laws	$p \wedge \boldsymbol{T} \equiv p$ $p \vee \boldsymbol{F} \equiv p$
Domination Laws	$p \vee \boldsymbol{T} \equiv \boldsymbol{T}$ $p \wedge \boldsymbol{F} \equiv \boldsymbol{F}$
Idempotent Laws	$p \vee p \equiv p$ $p \wedge p \equiv p$
Double Negation Law	$\neg(\neg p) \equiv p$
Commutative Laws	$p \vee q \equiv q \vee p$ $p \wedge q \equiv q \wedge p$
Associative Laws	$(p \vee q) \vee r \equiv p \vee (q \vee r)$ $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$
Distributive Laws	$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
De Morgan's Laws	$\neg(p \wedge q) \equiv \neg p \vee \neg q$ $\neg(p \vee q) \equiv \neg p \wedge \neg q$
Absorption Laws	$p \vee (p \wedge q) \equiv p$ $p \wedge (p \vee q) \equiv p$
Negation Laws	$p \vee \neg p \equiv \boldsymbol{T}$ $p \wedge \neg p \equiv \boldsymbol{F}$

Logical Equivalences Involving Implications (A Special Case)

Name	*Equivalence*
Implication Law (in terms of ∨)	$p \to q \equiv \neg p \vee q$
Contrapositive Law	$p \to q \equiv \neg q \to \neg p$
Negation Law in Implication (in terms of ∧)	$\neg(p \to q) \equiv p \wedge \neg q$
Distributive Law in Implication	$p \to (q \wedge r) \equiv (p \to q) \wedge (p \to r)$ $p \to (q \vee r) \equiv (p \to q) \vee (p \to r)$ $(p \vee q) \to r \equiv (p \to r) \wedge (q \to r)$ $(p \wedge q) \to r \equiv (p \to r) \vee (q \to r)$

Logical Equivalences Involving Bi-Implications (A Special Case)

Name	*Equivalence*
Equivalence Law	$p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$ $p \leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$ $p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q$
Negation Law in Bi-Implication	$\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$

Figure 3. Summary of logical equivalences used in the class

Table 6. Class discussion excerpt F

[128] Teacher:	What if I tell you that there are ten steps (in the proof)? I see ten steps ahead… But, I have not shown it.
[143] Student:	It is alright, (Sir) if you will perform your preferred method. Because we have our own way(s).
[144] Teacher:	How are you going to figure it out?
[145] Student:	For me, (Sir), it (the proof) must not immediately jump to 11.
[146] Teacher:	Why?
[147] Student:	We could not give the reason.

From Example 4 in the class discussion (Figure 4), the participants were tasked to substantiate each step. At this point, as they substantiate each step, they are also asked to discriminate the law they used from other similar-looking laws: Distributive law and De Morgan's law; Distributive law and Associative law; Commutative law and Idempotent law; Negation law and Domination law; and lastly, Identity law and Domination law (Table 7).

Example 4

Show that $\neg(\neg p \wedge q) \wedge (p \vee q)$ and p are logically equivalent by developing a series of logical equivalences.

Solution

$\neg(\neg p \wedge q) \wedge (p \vee q)$

$\equiv (\neg(\neg p) \vee \neg q) \wedge (p \vee q)$ *by De Morgan's Law*

$\equiv (p \vee \neg q) \wedge (p \vee q)$ *by Double Negation Law*

$\equiv p \vee (\neg q \wedge q)$ *by Distributive Law*

$\equiv p \vee (q \wedge \neg q)$ *by Commutative Law*

$\equiv p \vee F$ *by Negation Law*

$\equiv p$ *by Identity Law* ■

Summary of Logical Equivalences (A Special Case)	
Name	***Equivalence***
Identity Laws	$p \wedge T \equiv p$ $p \vee F \equiv p$
Domination Laws	$p \vee T \equiv T$ $p \wedge F \equiv F$
Idempotent Laws	$p \vee p \equiv p$ $p \wedge p \equiv p$
Double Negation Law	$\neg(\neg p) \equiv p$
Commutative Laws	$p \vee q \equiv q \vee p$ $p \wedge q \equiv q \wedge p$
Associative Laws	$(p \vee q) \vee r \equiv p \vee (q \vee r)$ $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$
Distributive Laws	$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
De Morgan's Laws	$\neg(p \wedge q) \equiv \neg p \vee \neg q$ $\neg(p \vee q) \equiv \neg p \wedge \neg q$
Absorption Laws	$p \vee (p \wedge q) \equiv p$ $p \wedge (p \vee q) \equiv p$
Negation Laws	$p \vee \neg p \equiv T$ $p \wedge \neg p \equiv F$

Figure 4. The fourth example of propositional equivalence

Table 7. Class discussion excerpt G

[335] Teacher:	So, now, how sure are you that this falls under distributive (property) and not associative?
[336] Student:	Based on what A*** had said, they do not have the same logical operators. There is, uhm, … disjunction, then, conjunction. Then, the common (terms) are brought out, which is, uhm, the logical operator of disjunction that is why the disjunction of *p* is brought out.
[357] Teacher:	How do I know if I will be using idempotent and not commutative? Or, commutative and not idempotent? How do they differ?
[358] Student:	… commutative (property) will be used if they are not the same. For example, the negation of *q* and *q*, they are different, that is why we will be using commutative. For idempotent, if the two (propositions) are the same, *p* or *p*, then that is the time we will be applying it.
[360] Student:	In commutative law, their difference between the idempotent, is uhm…, there is one variable used for idempotent… but for commutative laws, two variables are involved.
[373] Teacher:	What is the difference between negation and domination?
[390] Student:	I think Sir, there is no negation sign in domination laws.

The class discussion shows that the participants performed a *substantiation-type exploration routine* via *classroom discourses* utilising *object-driven words*. At this point, the construction-type exploration routine is not yet evident in the discourses. Example 5 (Figure 5) was specified to examine this routine. This example was adapted from the module. During the class discussions, participants were asked to close their notes and construct new narratives that led to the same goal. During the class discussion on Example 5, the construction of proofs varied, i.e., from what is offered in the module. The students confirmed this when asked about the possibility of having various solutions (Table 8). Thus, the construction routine is evident. The students' narratives from the example are endorsable, along with the mediators used.

Example 5

Show that $\neg(p \vee (\neg p \wedge q))$ and $\neg p \wedge \neg q$ are logically equivalent by developing a series of logical equivalences.

Solution $\neg(p \vee (\neg p \wedge q)) \equiv \neg p \wedge \neg q$

$\neg(p \vee (\neg p \wedge q))$

$\equiv \neg[(p \vee \neg p) \wedge (p \vee q)]$	by Distributive Law
$\equiv \neg[(\mathbf{T}) \wedge (p \vee q)]$	by Negation Law
$\equiv \neg[(\mathbf{T} \wedge p) \vee (\mathbf{T} \wedge q)]$	by Distributive Law
$\equiv \neg[(p \wedge \mathbf{T}) \vee (q \wedge \mathbf{T})]$	by Commutative Law
$\equiv \neg(p \vee q)$	by Identity Law
$\equiv \neg p \wedge \neg q$	by De Morgan's Law

Summary of Logical Equivalences (A Special Case)

Name	***Equivalence***
Identity Laws	$p \wedge \boldsymbol{T} \equiv p$ $p \vee \boldsymbol{F} \equiv p$
Domination Laws	$p \vee \boldsymbol{T} \equiv \boldsymbol{T}$ $p \wedge \boldsymbol{F} \equiv \boldsymbol{F}$
Idempotent Laws	$p \vee p \equiv p$ $p \wedge p \equiv p$
Double Negation Law	$\neg(\neg p) \equiv p$
Commutative Laws	$p \vee q \equiv q \vee p$ $p \wedge q \equiv q \wedge p$
Associative Laws	$(p \vee q) \vee r \equiv p \vee (q \vee r)$ $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$
Distributive Laws	$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
De Morgan's Laws	$\neg(p \wedge q) \equiv \neg p \vee \neg q$ $\neg(p \vee q) \equiv \neg p \wedge \neg q$
Absorption Laws	$p \vee (p \wedge q) \equiv p$ $p \wedge (p \vee q) \equiv p$
Negation Laws	$p \vee \neg p \equiv \boldsymbol{T}$ $p \wedge \neg p \equiv \boldsymbol{F}$

Figure 5. The fifth example of propositional equivalence

Table 8. Class discussion excerpt H

[406] Teacher:	Okay, let's look (at this example). Uhm, I have not shown any succeeding steps. I want you to think. Can we produce a solution right now that is different from our handout?
[407] Student:	It is possible, Sir.
[412] Teacher:	Okay, we are sure of it. Every student may have different (solutions).

From the FGD (Table 9), when asked about their mechanisms and activity structures in establishing propositional equivalences, Respondent A, in the individualisation of the collective, adapts to monitoring the metacognitive process. The metalevel is informed by the laws (i.e., legalities or object-level rules). Respondent B is operating more on the control metacognitive process. She regulates all possible laws that can be applied to the attainment of the goal (i.e., establishing propositional equivalences) by focusing on what is to be proved. Respondent C is more on the monitoring metacognitive process since the metalevel and other activity structures rely on the results of the truth tables (i.e., the object-level rules). Respondent D operates more on the monitoring since the set of so-called object-level rules informed the metalevel.

Table 9. FGD excerpt A

[Fgd27:34] A:	It's like, looking whether the solution is legal (acceptable).
[Fgd27:53] B:	I am looking for, the first one, in which it will lead to the logically equivalent statement. The end-product… I will look for the possible law that I can apply at the end.
[Fgd28:45] C:	I started with the truth, Sir. In the given, Sir, I checked if it is a tautology… (i.e., referring to the previous task). If it is not a tautology, I will not continue proving.

[Fgd29:13] D:	Sir, my work is like the PEMDAS. I am starting with (the terms) inside the parenthesis, the grouping symbols. Although the process is long, we will still reach the end. It will be proven at the end.

In terms of routines, seeking the ‘mathematical’ legality of activity structures, such as following the order of operations of logical operators and particular laws of logical equivalences, along with the checking of the truth values of the given propositional forms through the truth tables, falls in the *substantiation type exploration routines*. Inversely, focusing on the end goal, i.e., what is to be proved, energies the *construction routines* to produce a new narrative. As confirmed through the teacher-researcher class discussions, student works (Figure 6) and the FGD (Table 10), construction routines undeniably helped them be more object-driven in their word use and operate in a more literate discourse.

Table 10. FGD excerpt B

[Fgd36:35] B:	I learned from logical equivalences are the, uhm…, like what you have taught last time, (Sir), when in commutative, one propositional form is like representing a proposition. So, for example, in the table, there are only *p* or *q*, but in the solution or given, it can be a propositional form, and can be used like a single proposition. It may look complex because even me, I got confused in that (table).

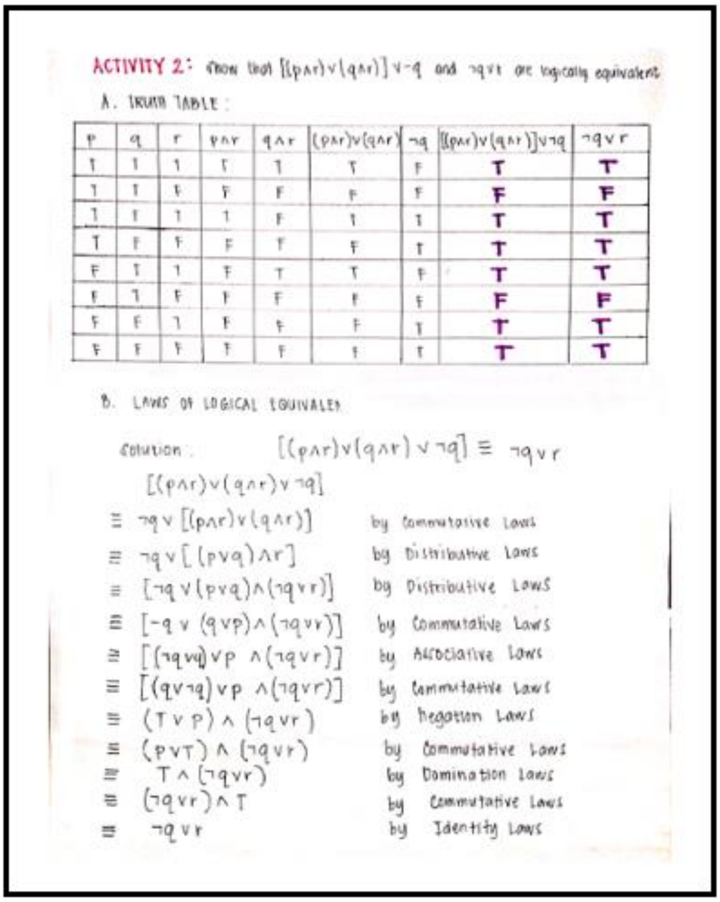

Figure 6. Student work

5. Conclusion and recommendations

This study examined the students’ discourses, in light of the commognitive, mainly the words used, visual mediators, endorsed narratives and routines, from participationist’s learning views. The words used in the discourse in defining the logical equivalence concept are, generally, routine-driven.

The participants' words used advanced to phrase-driven use upon describing the propositional variables. A commognitive conflict, via the occurrence of the word 'propositional statement', served as access to the new discourse on propositional form concept. The communalisation of the individual presented a ritualised classroom discourse, indicative of developing object-driven word use, showing ritual discursive routines. In defining the relevance of the substantiation and construction routines in establishing propositional equivalences, the conversation was somewhat colloquial to classroom discourse with some phrase-driven word use and not highly literate discourse of massive mathematical objects, providing clear wide-ranging perspectives. In both the construction and substantiation routines, the narratives produced during the communalisation are endorsable, along with the visual mediators used. In the individualisation of the collective, looking for the legality of activity structures, such as following the order of operations of logical operators and specific laws of logical equivalences, along with the checking of the truth values of the given propositional forms through the truth tables, falls in the substantiation type exploration routines, indicative of monitoring metacognitive process. Inversely, focusing on what is to be proved energies the construction routines, indicative of a control metacognitive process. Providing construction routine opportunities helped participants be more object-driven in their word use and operate in a more literate discourse. This study recommends to future researchers to explore and examine not only the content knowledge but also the pedagogical knowledge of mathematics pre-service teachers in the commognitive and a similar study on other pure mathematics courses within the Bachelor of Secondary Education mathematics curriculum.

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