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## Logic and heuristic in education

**Aslanbek Naziev\***, Department of Mathematics and Methods of Mathematics Teaching, Ryazan State University, Ryazan 390000, Russia

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### Abstract

More than once were suggested that the results of the field of logic appear to be useful at the checking of ready-made proofs but refused to serve when one has to discover evidence; although some very reputable mathematicians supported this view, we allow ourselves to disagree with it. Every time they offer arguments for this point of view (and very often it speaks without any arguments), they demonstrate a 'frontal', straightforward interpretation of the results of logic. However, such an interpretation of the interrelations between logic and heuristic is not the only possible one. On the contrary, other interpretations possible, in which logic and heuristics are two sides of the single process of learning. In our paper, we consider both mentioned uses of the logic.

**Keywords:** Intuitive evidence, logical proof, heuristic advices.

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\*ADDRESS FOR CORRESPONDENCE: **Aslanbek Naziev**, Department of Mathematics and Methods of Mathematics Teaching, Ryazan State University, Ryazan 390000, Russia. *E-mail address:* [a.naziev@365.rsu.edu.ru](mailto:a.naziev@365.rsu.edu.ru) / Tel.: +79106458189

## 1. Purpose of the article

Many reputable psychologists and educators believe that heuristics and logic are in no way related to each other. The creative process of 'discovery' of the solution of the problem they connect with heuristic and with logic—the verification of the finished solution for the truth, rigor and validity of each step of the solution. The most prominent supporter of this position was George Polya. In his book Polya (1981), he denies any possibility of using logic in the search for a solution to a problem. According to him, 'logic is the lady at the exit of the supermarket who checks the price of each item in a large basket whose contents she did not collect'. Similar thought expressed even earlier Henry Poincare: 'it is by the logic that we prove but the intuition that we discover' (Poincare, 1914, Ch. 2, no. 9, p. 129).

However, looking at heuristic and logic as two completely incompatible domains is not the only possible one. On the contrary, other interpretations are possible, in which the logic is capable not only of 'processing' discoveries obtained with the help of heuristics but also itself may lead to discoveries, that is, it possesses heuristic capabilities.

To understand this, sufficient to think what does it mean to make a discovery? This means not only to talk about some new idea, proposal, but also to give a reasoning that convinces the truth of this proposal, and, thus, to prove the validity of discovery since nothing is open until proven. It means that the logic takes the most direct part in the process of opening a new one.

In our paper, we first show on examples how intuition can lead to wrong solutions and logic helps clarify the situation and correct the mistakes. After that, we demonstrate several heuristic features of logic.

## 2. Main results

### 2.1. Pitfalls of intuitive evidence

One of the divisions of mathematics where logic is supposed to play the vital role but constantly ignored is the school geometry. The reason is quite understandable although not always noticed. It is in a false sense of clarity, generating the illusion of evidence in geometry. Bring the examples that support these statements.

The first of these examples we took from a book on geometry (Adler, 1958).

**Example.** Every triangle is isosceles.

**Proof.** Let  $ABC$  be a triangle as shown in Figure 1. We want to prove  $CA = CB$ . Let  $D$  be the intersection of the perpendicular bisector of  $AB$  and the internal bisector of angle  $ACB$ . Let  $DE \perp AC$  and  $DF \perp CB$ . It is easy to see that  $\triangle CDE \cong \triangle CDF$  and  $\triangle BDF \cong \triangle ADE$ . Hence  $CE + EA = CF + FB$ , i.e.,  $CA = CB$ . QED!

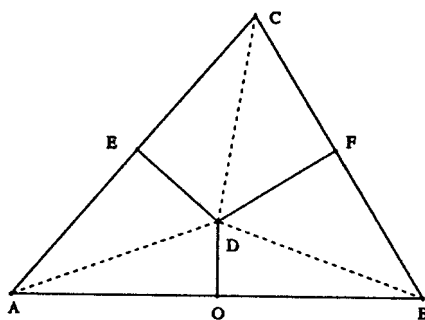


Figure 1. Every triangle is isosceles

This example reveals the shortcomings of the teaching mathematics in isolation from logic. It follows this guideline: 'Allow your hand to draw everything that it pleases, look at what happened, and base the solution of the problem on the fantasies of your hand'!

The example is remarkable, but it is *invented*, while the following example is taken from *real life*, namely, from a school textbook written by the famous Russian mathematician and mathematics didact.

**Example** (Pogorelov, 2013, § 5, **Problem 3**). Prove that if a diameter of a circle passes through the midpoint of the chord, then the diameter is perpendicular to the chord.

The author provides the problem with the solution based on the Figure 2.

However, the triangle  $AOB$  will not exist if the chord passes through the centre of the circle, and the diameter, in this case, is not necessary perpendicular to the chord (draw the corresponding picture!). So, this famous mathematician and mathematics educator builds the above solution not on axioms, definitions and theorems, not on *logic*, but also on extra considerations such as a figure drawn by his hand!

This mistake is, especially, astonishing because, first, it is evident, and, second, even in Euclid's *Elements* one can see the correct statement: if a diameter of a circle passes through the midpoint of a chord that is not a diameter, then the diameter is perpendicular to the chord. We see the clearest example of the fact that even an *outstanding* mathematician and even in the *simplest* situation can come to the wrong conclusions if he relies not on logic but on intuitive evidence!

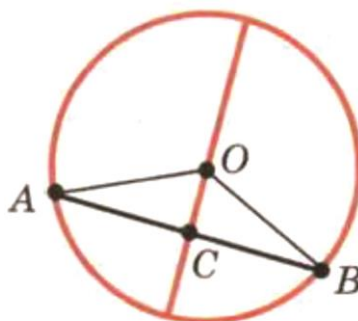


Figure 2. A circle, a chord and a diameter

## 2.2. Heuristic features of logic

**To begin with heuristic features of logic, take, for example, the rule of detachment, according to which whenever the sentence  $\varphi$  and the implication 'if  $\varphi$ , then  $\psi$ ' be theorems, also the sentence  $\psi$  is a theorem. This rule suggests at least three heuristic ways of searching for the proof.**

- 1) We have to prove the proposition  $\psi$ . Do it directly not turning well. However, it is clear that the proposition  $\varphi$  is a theorem. Maybe, the implication 'if  $\varphi$ , then  $\psi$ ' also is a theorem? If we could establish it, then, applying the rule of detachment, we would have the proof for  $\psi$ .
- 2) We have to prove the proposition  $\psi$ . Do it directly not turning well. However, we see (or can establish) that the implication 'if  $\varphi$  then  $\psi$ ' is a theorem. Would not it be true also for  $\varphi$ ? In such a case, again with the help of the rule of detachment, we would obtain the proof for  $\psi$ .
- 3) We need to prove the proposition  $\psi$ . Do it directly not turning well. Are there known theorems of the form 'if ..., then  $\psi$ '? Here is such a theorem: 'if  $\varphi$ , then  $\psi$ '. Does not  $\varphi$  be a theorem?

Even the most primitive at the first glance results of logic have important applications in training. Consider two such results:

- 1) the false premise law, according to which the implication with an invalid premise is undoubtedly correct, and
- 2) the law of correct conclusion according to which the implication with the correct conclusion is undoubtedly true.

These laws prove to be extremely helpful in eradicating of one widespread error in the proofs. We mean an error consisting in that instead of inference the sentence, say  $\phi$ , which one has to prove, from some faithful, they deduce some correct sentence from  $\phi$  (that they have to prove). Both of above-mentioned laws make obvious the fallacy of this ‘way of reasoning’.

Indeed, deducing from  $\phi$  some correct sentence  $\psi$ , we thereby make sure that the implication ‘if  $\phi$ , then  $\psi$ ’ is fair. What does this say about  $\phi$ ?—Nothing for: (1) the implication with a false premise is true with any conclusion so that it is possible that we have been able to deduce  $\psi$  out of  $\phi$  simply because that  $\phi$  is wrong; (2) the implication with a correct conclusion is valid for any premise, including incorrect. We see that underlining the role of the two logical laws in this question would help those who correctly recognised them, to find in the future correct proofs instead of the invalid that seem to them valid.

Very effective for solving problems is also the law of expression of disjunction through implication: ‘Whatever the proposals  $\phi$  and  $\psi$ ’, there is an equivalence

$$(\phi \vee \psi) \leftrightarrow (\neg\phi \rightarrow \psi)$$

Most useful this law is for proving disjunctions.

**Example.** Prove that for any numbers, say,  $a$  and  $b$ , if  $ab=0$ , then  $a=0$  or  $b=0$ .

**Proof.** Let  $ab=0$ . Under this assumption, we need to prove the disjunction  $a=0$  or  $b=0$ . However, how to prove it? We may begin to prove the first term, but the second one is probably true. We may begin to prove the second term, but the first one is probably true. We can get out of this difficulty as follows:

Either  $a=0$  or  $a \neq 0$ .

If  $a \neq 0$ , then the first term of the proving disjunction takes place. If  $a \neq 0$ , then there exists  $c$  such that  $ca=0$ . Multiplying both sides of the equation  $ab=0$  by  $c$ , we get  $c.ab=c.0$ , that is  $b=0$ .

However, we can do it even easier. Knowing the law of expression of a disjunction through the implication, we have no longer to begin with ‘either-or’; we can immediately proceed to the proof of the implication of ‘if  $a \neq 0$ , then  $b=0$ ’.

Hence, the very useful advice:

‘If you need to prove a disjunction, try to derive one of its members from the negation of the other’.

Thus, mathematical logic advises how to search for a solution to a problem, that is, it produces heuristics, and, therefore, participates not only in the process of proof but also in the process of discovery.

Let us bring a less evident example. Consider the tautology

$$(\phi \rightarrow (\psi \leftrightarrow \chi)) \leftrightarrow (\phi \wedge \psi \leftrightarrow \phi \wedge \chi)$$

(the so-called ‘distributive law of implication relative to equivalence’, or ‘of conditional relative to bi-conditional’). One can interpret this tautology primitively: this propositional scheme has true value

under all the values of its propositional variables. However, there also exist other interpretations. The conjunction ' $\phi \wedge \psi$ ' is the same thing as the system

$$\begin{cases} \phi, \\ \psi, \end{cases}$$

From this point of view, the tautology under consideration tells us that the systems

$$\begin{cases} \phi, \\ \psi \end{cases} \text{ and } \begin{cases} \phi, \\ \chi \end{cases}$$

are equivalent precisely when  $\psi$  and  $\chi$  are equivalent under the presupposition  $\phi$ . In other words, knowing that this scheme is a tautology teaches us to look at the system in the whole. It shows that to the equivalency of the two written systems is not necessary that their second members were equivalent absolutely, it is sufficient that their equivalence follows from the presupposition  $\phi$ . It suggests us, when we transform someone member of the system, use other members as assumptions. It is very useful advice. Such applications have every logical law; one has only to try to see them.

Let us illustrate the application what was said to with the following

**Example.** Solve the system:

$$\begin{cases} x + y = 3z \\ x^2 + y^2 = 5z \\ x^3 + y^3 = 9z \end{cases}$$

This system we took from the book (Shikhanovitch, 1965, pp. 328–332), where this system is named 'real', and its solution occupies almost five pages. Above-mentioned considerations permit to go into one page of the same size.

Solution. Let

$$\begin{cases} x + y = 3z \\ x^2 + y^2 = 5z \end{cases}$$

$$\text{Then } 5z = (x + y)^2 - 2xy = 9z^2 - 2xy ,$$

whence

$$2xy = 9z^2 - 5z ,$$

from where,

$$xy = \frac{1}{2}z(9z - 5) .$$

Therefore,

$$x^3 + y^3 = (x + y)(x^2 - xy + y^2)$$

$$= 3z \cdot \left(5z - \frac{1}{2}z(9z - 5)\right)$$

$$= \frac{9}{2}z^2(5 - 3z).$$

By force of this, the last equation of the system precisely means that

$$\frac{9}{2}z^2(5 - 3z) = 9z$$

i.e., that

$$9z = 0 \text{ or } z(5 - 3z) = 2 ,$$

i.e., that

$$z = 0 \text{ or } z^2 - 5z + 2 = 0 .$$

Furthermore, because

$$3z^2 - 5z + 2 = 3z^2 - 3z - 2z + 2$$

$$= 3z(z - 1) - 2(z - 1)$$

$$= (z - 1)(3z - 2),$$

then  $3z^2 - 5z + 2 = 0$  precisely when  $(z - 1)(3z - 2) = 0$ , i.e., when  $z = 1$  or  $z = 2/3$ .

Therefore, under the presupposition, with which we began our solution, the last equation of the system fulfils precisely when the disjunction 'z = 0 or z = 2/3 or z = 1' do. This statement means that the system under consideration is equivalent to the disjunction of the three following systems:

$$\begin{cases} x + y = 3z, \\ x^2 + y^2 = 5z, \\ z = 0, \end{cases} \begin{cases} x + y = 3z, \\ x^2 + y^2 = 5z, \\ z = \frac{2}{3}, \end{cases} \begin{cases} x + y = 3z, \\ x^2 + y^2 = 5z, \\ z = 1. \end{cases}$$

Solve these systems in turn, proceeding as above.

1) Let  $z = 0$ . Then  $x^2 + y^2 = 5z$  if and only if  $x^2 + y^2 = 0$ , that is when  $x = y = 0$ . It is obvious that these numbers fulfil the first equation.

2) Let  $z = 2/3$  and  $x + y = 3z$ , i.e.,  $x + y = 2$ . Then the second equation of the system means that  $x^2 + (2 - x)^2 = \frac{10}{3}$ , i.e.,  $2x^2 - 4x + 4 = \frac{10}{3}$ , i.e.,  $3x^2 - 6x + 1 = 0$ , i.e.,  $x = 1 \pm \frac{\sqrt{6}}{3}$  and so,  $y = 1 \mp \frac{\sqrt{6}}{3}$ .

3) Let  $z = 1$  and  $x + y = 3z$ , i.e.,  $x + y = 3$ . Then the second equation of the system means  $x^2 + (3 - x)^2 = 5$ , i.e.,  $x^2 - 3x + 2 = 0$ , i.e.,  $x = 1$  or  $x = 2$  and so,  $y = 2$ , accordingly,  $y = 1$ .

**Answer.** The system has precisely the following five solutions:

$$\left\{ \begin{array}{l} x = 0, \\ y = 0, \\ z = 0, \end{array} \right. \left\{ \begin{array}{l} x = 1 \pm \frac{\sqrt{6}}{3}, \\ y = 1 \pm \frac{\sqrt{6}}{3}, \\ z = \frac{2}{3}, \end{array} \right. \left\{ \begin{array}{l} x = 1, \\ y = 2, \\ z = 3, \end{array} \right. \left\{ \begin{array}{l} x = 2, \\ y = 1, \\ z = 1. \end{array} \right.$$

### 3. Discussion

In this paper, we saw that Mathematical Logic (ML) not only can participate in the process of verification of validity results and correctness of arguments but also provide fruitful heuristic suggestions for investigations.

The question naturally arises, why such outstanding mathematicians and mathematics educators as Poincare and Polya deny such a possibility. What about Poincare, he merely could not know the body of results of Mathematical Logic, obtained since 1910. Maybe, Polya also not knew these results because they lie outside of the domain of his interests. But there exists one more reason, more valuable in our opinion.

In the book Hao (1974), its author wrote:

‘If mathematical logic were a little less pure, perhaps it could assist a mathematician to learn some alien branch of mathematics. In its present aloof form, however, training in mathematical logic is neither necessary nor likely to speed up the pursuit of other branches of mathematics’ (p. 229).

These words was written in 1959 (and repeated in the cited book in 1974) but until now hardly one can find a textbook on ML to which these words are not applicable. Namely this tradition of teaching ML leads to the fact that there exist so many people thinking that ML and heuristic has nothing in common.

### 4. Conclusion

We saw that even the simplest results of ML have important heuristic parallels. But, this side of teaching ML remains until now underestimated. In view of this, we come to the following recommendation.

### 5. Recommendation

We recommend the authors of textbooks in ML to accompany the training of this subject by heuristic considerations similar to those which we demonstrated in our paper. This change will help ML to become ‘little less pure’ and permits ML to purchase more ‘friendly interface’ instead of the mentioned by Hao Wang ‘aloof form’.

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