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Comparative study and evaluation of dominant external representational systems in mathematics education

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Abstract

In this theoretical study, initially, we propose and argue in favour of sub-dividing the *external representations* into the following two major categories: the *self-constructed external representations and the hetero-constructed external representations*. We then link this distinction to the general meaning of representations. Some researchers mistakenly use *representational mode, representational method and representational system* as synonymous or identical concepts thus often confusing. This is the reason why we put effort in outlining the 'fluid' conceptual boundaries of these concepts by highlighting the relation between them. Hence, we outline and compare the dominant categorisations of external representations proposed by various researchers while examining the necessity for their improvement and their extension.

Keywords: External representational systems, self-constructed external representations, hetero-constructed external representations, representational element, representational mode, representational method, representational code.

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1. Introduction

External representations are known to be an integral part of Mathematics education as they are compulsory in teaching mathematical concepts via symbols and diagrams. In addition, external representations, as they are tangible, can be subject to direct observation and they usually externalise the mean learners utilise to understand Mathematics. Therefore, external representations are significant when they come to Mathematics education. For a better understanding of the significance of the external representations, we consider it wise to sub-divide the external representations into the following two major categories: the *self-constructed external representations and the hetero-constructed external representations.*

We argue that the *self-constructed external representations* are identical to the external representations produced by the learners themselves (students, pupils and trainees). The role of the self-constructed external representations has more to do with recording the internal representations while providing an opportunity to examine the communication channels used to approach or work with mathematical thoughts, ideas or concepts.

In conclusion, the value of self-constructed external representations lies in the fact that 'more or less, when handled or modified, the self-constructed external representations can indicate the internal representations which they relate to' (Janvier, Girardon & Morand, 1993).

We also construct the term '*hetero-constructed external representations*' to name a variety of representations on mathematics tasks given to learners (by teachers or via textbooks or via mathematics software) for them to work on/with them.

Both self-constructed and hetero-constructed external representations may play a significant role in the problem-solving process given the fact that in the educational process, clarifying and highlighting the mathematical concepts that are taught to learners are likely to be facilitated by extensive exploration and utilisation of the representations.

Below, we first present conceptual distinctions on representations in mathematics education and put forward proposals on how to extend them. We then present the main categories into which researchers Bruner (1966), Lesh, Post and Behr (1987) and Nakahara (2008) divided the representations while analysing the characteristics of these categories. We then outline the intersections that the above these categorisations have in common.

1.1. Basic conceptual distinctions

Nakahara (2008) attempts to proceed with a distinction between the following terms concerning the representational systems: «*representational method*», «*representational mode*» and «*representational system*». He argues that:

The representational method represents a clear mathematical expressions (i.e., '3+2' or 'add 3 plus 2'). The authors of this study use the term '*representational form*' as a synonym and instead of the term 'representational method'.

The representational mode is a set of specific representational methods/forms that are categorised by a specific criterion (i.e., 'representation using symbols', 'representation using numbers' and 'representation using geometrical shapes').

The representational system systematically organises the mutual relations between the representational modes (Nakahara, 2008).

In this general distinction between the afore-mentioned terms by Nakahara, the term of *«representational code»* is also introduced by the authors and its definition *is the combination of many representational systems*, namely, a *hyper-system* that *combines ontological and epistemological beliefs on representations*. The representational code is the intersection of mathematical representations, dominant social structures and values about Mathematics. For example, Leone's Burton (2004) model[1] – addressing four epistemological challenges – offers specific ways of managing representational codes. Extensive ways of using representational codes could derive, among other philosophical and epistemological theories, from: a) the Principles of Logicism, b) the Principles of Formalism, c) the Principles of Constructivism, d) the Principles of Intuitionism and e) the Principles of Phenomenology.

The authors have also added another basic term, which is related to the mathematical representations, this being *'representational element'.* The representation element is the most simple representation and it is regarded as a self-constructed or hetero-constructed representation, which causes a dominant sensory stimulus (visual, tactile, kinaesthetic, audible, verbal) to learners, or a multi-sensory one, e.g., 1, x, y, %, ….

In Figure 1, it is presented an icon for the relation between the *representational element*, the representational *method/form*, the representational *mode*, the representational *system* and the representational *code*.

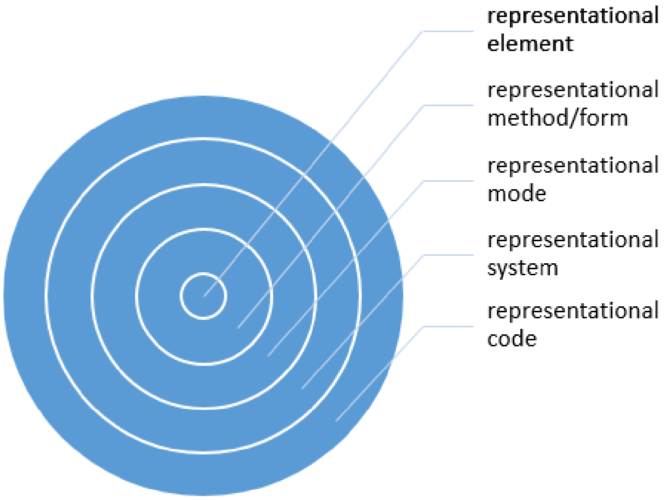

Figure 1. The relationship between extended representational terms, which are related to the mathematical representations created by Moskofoglou and Vamvouli (2019)

2. The Bruner representational systems model enactive-iconic-symbolic ('EIS')

Bruner (1966) as cognitive psychologist focused on students' knowledge and representational thoughts. It is well known that Bruner's three stages of the representational systems are:

- Enactive representation («E»)

  External reality is conceived via action, impersonation and objects' manipulation.

- Iconic representation («I»)

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1 This epistemological model presented the process of *«coming to know Mathematics»* and in its revised version (i.e., Leone Burton, 2004 in Moskofoglou-Chionidou, 2013) it addresses the following four epistemological challenges:

1. From Objectivity in Mathematics to a Socio-cultural Structure
2. From Homogeneity to Heterogeneity or Polysemy
3. From Impersonal to Holistic
4. From Discontinuity or Segmentation to Cohesion and Conceptual Interconnection

Internal spiritual images depict the outside world. However, it would be wise to use a variety of supervisory means during the teaching process since the external object and the corresponding internal symbol cannot be completely separated.

- Symbolic representation («S»)

The learner usually represents the external reality using abstract symbols such as words, mathematical symbols and signs.

The so-called «EIS Principle» – its name corresponding to the initials of Bruner's three representational systems (Enactive-Iconic-Symbolic) has been used both for early teaching of mathematical concepts and for educational instructions for pupils with learning difficulties). Above all, the «EIS Principle» has played a significant role in the so-called 'Modernisation Movement' in mathematics' education.

Although these three representational modes are considered as very important for the thorough comprehension of Mathematics, some researchers have reduced them to two categories: linguistic and non-linguistic representational systems (Marzano, 2004; Marzano, Pickering & Pollock, 2001). Other researchers have added additional categories such as real life experiences, manipulative models, pictures or diagrams, spoken words and written symbols (Lesh, Landau & Hamilton, 1983). Two of these models are presented below.

3. The Lesh, Post and Behr representational systems model

Lesh et al. (1987), who have been researching since 1983, expanded the Bruner representational systems model and classify the representational systems in the following five categories:

- *Experience-based 'scripts'* in which knowledge is organised around 'real world' events and serving as general frameworks that will facilitate interpretation and problem solving.
- Manipulative models (i.e., *Cuisenaire rods, arithmetic blocks and number lines*) for which the system's single components are insignificant. However, the relationships and functions, which result from managing the system's components, can be adapted to many everyday situations.
- Pictures or diagrams – static/figural models, which can be internalised as 'images' similar to the Manipulatable models.
- Spoken languages, including specialised sub-languages associated with fields such as the field of logic.

Written symbols, which may include specialised sentences and phrases [$x + 3 = 7$, A′U B' = (A∩B)], as well as regular sentences and phrases of the spoken language.

In fact, the afore-mentioned five categories of the representational systems are interconnected in any given combination among them [Lesh et al., (1987)].

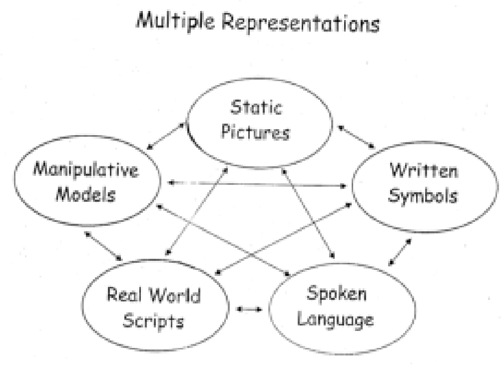

Figure 2. The Lesh, Post and Behr representational systems mode l (1987)

When comparing the Lesh, Post and Behr representational systems model to the Bruner representational systems model, we realise that the former is not a linear model. On the contrary, the Lesh, Post and Behr representational systems model includes all possible combinations and reciprocal transformations between the five different representational modes suggested by Lesh et al. (1987). At the same time, we may make a variety of transformations within the Lesh, Post and Behr representational systems model. Researchers Lesh, Post and Behr divide Bruner's 'enactive representation' into two categories: 'real scripts' and 'manipulative models'. The reason for this sub-division is that these two categories differ significantly in the degree of abstraction and accuracy in mathematics learning.

4. Nakahara's representational systems in mathematics' education (2008)

Nakahara (2008) – starting with Lesh's et al. study – has analysed and reviewed many previous researches and suggests *a broader coordination for representational modes* in Mathematics by classifying them in the following five categories:

S2. Symbolic Representation used in mathematical notation/mathematical writing (i.e., numbers, letters and symbols),

S1. Linguistic Representation used in maths problems as well as in colloquial language.

- *I. Illustrative Representation*, which utilises illustrations, shapes, charts, etc.
- *E2. Manipulative Representation*, which includes teaching aids (i.e., use of tangible material) that work by adding the dynamic functionality of the objects that are either artificially constructed or modelled.
- *E1. Realistic Representation* based on factual situations and objects.

We observe Nakahara's examples for each representational mode in Figure 3

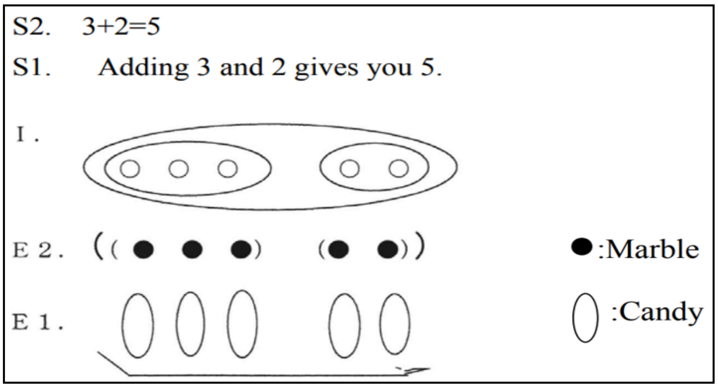

Figure 3. Nakahara's examples for each representational mode

A comparison between the above categories and Bruner's 'EIS' Principal shows that Bruner's Enactive Representational System is divided, by Nakahara, into two categories of representational modes: the 'realistic representation' and the 'manipulative' one. Bruner's symbolic representational system is also divided into two representational modes: the 'linguistic representation' and the 'symbolic' one.

Furthermore, when comparing the above to Lesh's et al. (1987) representational systems we see that 'spoken languages' and 'real scripts' are eliminated, and 'written symbols' are divided into 'linguistic representation' and 'symbolic representation', whereas manipulative models are divided into 'manipulative representation' and 'realistic representation'.

Table 1. Comparative table of dominant representational systems in mathematics education

	Bruner (1966)	Lesh et al. (1987)	Nakahara (2008)
Representational Modes	Enactive representation	Manipulative models	Manipulative representation realistic representation
	Symbolic representation	Written symbols	Symbolic representation linguistic representation
	Iconic representation	Static pictures	Illustrative representation
	-	Spoken language	-
	-	Real scripts	-

It seems that Nakahara's representational systems (2008) make better use than Bruner and Lesh's representational systems of Mathematics education because they incorporate specific features of the representational modes in Mathematics education. To be more specific:

1. The symbolic representational mode is governed by rules and is comprehensive and clear.
2. The linguistic representational mode, while contractual and lacking in content, is descriptive and offers a sense of familiarity.
3. The illustrative representational mode is rich both in terms of optics and intuition.
4. The manipulative representational mode is dynamic and somehow tangible and artificial.
5. The realistic representational mode is dynamic and extremely tangible and physical.

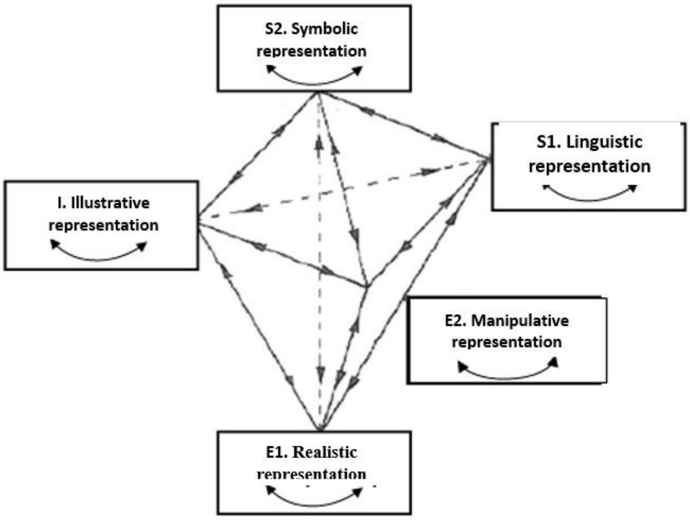

Figure 4. Nakahara's representational systems in mathematics' education (2008)

The types of reciprocal relations between the representational modes, as shown in *Figure 4*, are based on the following:

- The main sequence followed is E→I→S (bottom to top), as this represents the transition from specific representations to extremely abstract ones.
- Arrows are used to depict the mutual conversions between different representational modes as well as the mutual conversions within the same representational mode.
- Conversions between representational modes are referred to as 'translations'. This activity of translation seems to generate the mathematical thinking process, deepens the understanding process and promotes problem-solving processes.

When it comes to the use of the afore-mentioned representational modes, Nakahara (2008) deems it appropriate for junior grades learners to start with the 'realistic representations', given that in junior grades, many concepts and methods in mathematics result from solving realistic problems. At the end of junior grades, the concepts and the methods taught are usually depicted with mathematic symbols.

This is the point where Nakahara suggests using 'symbolic representations'. These two representational modes are associated with the 'enactive representation', the 'illustrative representation' and the 'linguistic representation'. Therefore, when using representational systems, it is advisable for junior grades pupils to first approach the realistic representations, hence illustrative and linguistic ones, and at the end of their lessons, students should be introduced to the symbolic representations.

5. Conclusions and recommendations

It is obvious that studying Nakahara's representational systems model (2008) complements and extends the bibliographically dominant the Lesh et al. (1987) model. As we have already mentioned above, the originality of the Nakahara model lays in the fact that it integrates and utilises the features of representational Mathematics education modes.

Furthermore, Nakahara's model does not merely make a categorisation but also proposes a hierarchical course of representational modes utilisation ranging from realistic representational modes to manipulative, pictorial, linguistic and finally to symbolic representational modes. It should be noted, however, that we are questioning the afore-mentioned model that is not a 'one-way street', which excludes other combinations for the transition from one mode to another. However, it is recommended as a good method for young pupils or for the introduction to new mathematical concepts. A similar attempt was made in the Bruner representational systems model with the utilisation of the 'EIS Principle'.

However, modern mathematical education utilises new representational modes (Novak, 2017), which cannot be included in Nakahara's model or Lesh, Post and Behr's model for example the Kinaesthetic Representational Mode, via the *'Maths In Motion'* program[2] or the *'Maths Dance'* programs,[3] etc. On the other hand, maybe it is safe to say that the Kinaesthetic Representational Mode could be classified under Bruner's 'manipulative representation', insofar, as it is through this representational mode that external reality is perceived, inter alia, via action. However, the Kinaesthetic Representational Mode offers a different dynamic to mathematical perception, which is why we reject its merging with any other model (Pourkos, 2016).

Moreover, these models fail to classify the multimodal representations that utilise more than two representational modes simultaneously under their categories. For example, where could one classify a digital educational application of 'Cuisenaire rods'? Would that be in Bruner's model, the Lesh, Post and Behr model or the Nakahara model? As it is known, 'Cuisenaire rods' are useful practical tools and, therefore, given their physical form, they should be classified under 'manipulative representations'. But, how will 'Cuisenaire rods' be categorised when converted to digital tools and used as such?

So it seems that the afore-mentioned existing dominant theoretical models for external representations fail to group all possible representational modes of mathematical concepts in a clear and functional way. The fact that these models are difficult to be applied when it comes to teaching practices – since they tend to generalise and, in some cases, they overlap – is another major drawback. For example, in the Lesh, Post and Behr model, the distinction between 'texts with empirical activities' and 'written symbols' is not clear as it is impossible to produce a text without the use of written symbols.

In conclusion, taking the above observations into account, we hereby propose the creation of a new inclusive representational systems model – theory in Mathematics teaching and learning processes. This model – theory should clearly answer research questions such as What are all the possible representational ways of specific mathematical concepts? By which criteria can they be grouped into

[2] Read more Retrieved November 17, 2019 from http://scico.gr/maths-in-motion/
[3] Read more Retrieved November 17, 2019 from http://www.mathsdance.com/

single categories? What are the possible relations between the representational modes? Is there a system one person should follow to move from one representational mode to another? How do representational modes constitute representational systems? Which representational codes make Mathematics meaningful and provide students with individualised and differentiative teaching and learning? and Which representational codes focus on anthropocentric teaching and learning processes that affect both the emotions and the physical and mental manifestations of the learner? This new model – theory, which will, hopefully, answer all these questions – is under construction and is currently experimented by our research team**.**

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